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SOLVABILITY OF A MIXED PROBLEM FOR A DEGENERATE SUB-DIFFUSION EQUATION IN A RECTANGULAR DOMAIN

О РАЗРЕШИМОСТИ СМЕШАННОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ ВЫРОЖДАЮЩЕГОСЯ СУБДИФФУЗИОННОГО УРАВНЕНИЯ В ПРЯМОУГОЛЬНОЙ ОБЛАСТИ

TO'G'RI TO'RTBURCHAKLI SOHADA BUZILADIGAN SUBDIFFUZIYA TENGLAMASI UCHUN ARALASH CHEGARAVIY MASALANING YECHILISHI

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Abstract

In the present paper, a mixed boundary value problem for a degenerate sub-diffusion equation in a rectangular domain is studied. By applying the method of separation of variables, a spectral problem for a space-dependent ordinary differential equation is derived. The existence of eigenvalues and eigenfunctions of the resulting spectral problem is established using the theory of positive differential operators. The solution to the problem is constructed in the form of a Fourier series. The convergence of the obtained series is proved, and the uniqueness of the solution is established by exploiting the completeness of the system of eigenfunctions.

Аннотация

В настоящей работе исследуется смешанная краевая задача для вырожденного уравнения субдиффузии в прямоугольной области. С использованием метода разделения переменных получена спектральная задача для обыкновенного дифференциального уравнения, зависящего от пространственной переменной. Существование собственных значений и собственных функций соответствующей спектральной задачи доказано на основе теории положительных дифференциальных операторов. Решение задачи строится в виде ряда Фурье. Доказана сходимость полученного ряда, а также установлена единственность решения на основе полноты системы собственных функций.

Annotatsiya

Ushbu maqolada to'g'ri to'rtburchak sohada buziladigan subdiffuziya tenglamasi uchun aralash chegaraviy masala o'rganilgan. O'zgaruvchilarni ajratish usuli yordamida fazoviy o'zgaruvchiga bog'liq bo'lgan oddiy differensial tenglama uchun spektral masala hosil qilinadi. Hosil bo'lgan spektral masalaning xos qiymatlari va xos funksiyalarining mavjudligi musbat differensial operatorlar nazariyasiga asoslanib isbotlangan. Masalaning yechimi Furye qatori ko'rinishida qurilgan. Olingan qatorning yaqinlashuvchiligi isbotlangan hamda yechimning yagonaligi xos funksiyalar sistemasining to'laligidan foydalanib asoslangan.

Key words: *fractional calculus, degenerate differential equation, mixed problem, Caputo fractional differential operator, spectral problem.*

Ключевые слова: *дробное исчисление, вырожденное дифференциальное уравнение, смешанная задача, дробный дифференциальный оператор Капуто, спектральная задача.*

Kalit so'zlar: *kasr tartibli hisob, buziladigandifferensial tenglama, aralash masala, kasr tartibli Kaputo differensial operatori, spektral masala.*

INTRODUCTION

Fractional partial differential equations have attracted considerable attention in recent years due to their ability to describe memory and hereditary effects inherent in many complex physical and biological processes. In particular, diffusion equations involving fractional-order time derivatives provide an adequate mathematical framework for modeling anomalous diffusion phenomena observed in heterogeneous and porous media, viscoelastic materials, heat conduction with memory, and transport processes in biological systems [1-3].

Among various definitions of fractional derivatives, the Caputo fractional derivative is especially convenient for physical applications, as it allows the formulation of initial conditions in terms of integer-order derivatives.

In many realistic models, diffusion processes are influenced not only by temporal memory effects but also by spatial inhomogeneities. This leads naturally to diffusion equations with spatially variable coefficients, which may vanish or become unbounded on certain subsets of the domain. Such equations are commonly referred to as *degenerate diffusion equations*. The presence of degeneration significantly affects the qualitative properties of solutions, including regularity, well-posedness, and the admissible forms of boundary conditions. Degenerate equations arise, for instance, in the mathematical modeling of heat conduction in materials with spatially varying conductivity, diffusion in porous or layered media, and vibration problems of elastic structures with nonuniform properties [4-6].

The analysis of degenerate partial differential equations with integer-order derivatives has a long history, and a variety of analytical techniques have been developed to study their solvability and spectral properties. However, the investigation of *fractional* diffusion equations with spatial degeneration remains relatively limited, especially in multidimensional settings and for mixed initial-boundary value problems. In particular, the interplay between the order of the fractional derivative and the degree of spatial degeneration poses substantial analytical challenges.

An important feature of space-degenerate equations is that the type and number of boundary conditions required for well-posedness depend essentially on the degree of degeneration. Unlike nondegenerate diffusion equations, where boundary conditions are prescribed uniformly along the boundary, degenerate problems may require fewer or different conditions on the degeneration set. A rigorous justification of such phenomena is crucial for both mathematical completeness and physical consistency of the models.

The present paper is devoted to the study of a mixed initial-boundary value problem for a time-fractional diffusion equation with spatial degeneration in a rectangular domain. The equation involves a Caputo fractional derivative of order $0<\alpha<1$ with respect to time and a second-order spatial operator whose coefficient vanishes on a part of the boundary, creating a line of degeneration. Special attention is paid to the dependence of boundary conditions on the degree of degeneration.

The analysis is based on the method of separation of variables, which leads to a weighted spectral problem associated with a degenerate differential operator. By employing variational techniques and the Friedrichs extension, we establish the existence of a discrete spectrum and construct a complete system of eigenfunctions. Using these eigenfunctions, the solution of the original problem is represented in the form of a Fourier series with coefficients expressed via Mittag-Leffler functions. Detailed estimates for the Fourier coefficients are derived to ensure convergence and justify term-by-term differentiation and integration.

As a result, sufficient conditions are obtained for the existence and uniqueness of classical or weak solutions, depending on the degree of degeneration. The results demonstrate how spatial degeneration influences both the functional framework and the formulation of boundary conditions, thereby extending classical solvability theory to a new class of fractional degenerate diffusion equations.

The structure of the paper is as follows. In Section 2, preliminary definitions and auxiliary results on fractional calculus and function spaces are presented. Section 3 is devoted to the analysis of boundary conditions depending on the degree of degeneration. In Section 4, the main mixed problem is formulated. Section 5 investigates the associated spectral problem and its properties. Section 6 contains estimates for Fourier coefficients and convergence results. Finally, Section 7 establishes the existence and uniqueness of solutions.

In a rectangular domain Ω such that $\Omega=\{(x,\ t): 0<x<1,\ 0<t\le T<\infty\}$, we consider the following equation

$$\partial_t^\alpha u(x,\ t)-\frac{\partial}{\partial x}\left(x^\beta \frac{\partial u}{\partial x}\right)=f(x,\ t),\ \ 0<\alpha<1, \tag{1}$$

where $\partial_t^\alpha u(x,t)$ is the Caputo differential operator of α fractional order [7], $\alpha,\beta\in R$ such that $\alpha\in(0,1)$, $\beta\in(0,2)$ and $\beta\neq 1$, $f(x,t)$ is a given function.

For equation (1), the line $x=0$ is the line of degeneration, and β is the degree of degeneration. So, the equation (1) can be considered space-degenerate partial differential equation. Our approach showed that the degree of degeneration β affects both the formulation and investigation of the problem. We will formulate and study a mixed problem for different degrees of degeneration in equation (1).

2. PRELIMINARIES

In this section, we give the definitions and some properties of the fractional differential operators and some function spaces that will be used throughout the paper.

2.1. Caputo fractional differential operator and Mittag-Leffler functions

Definition 2.1. *[7] Caputo fractional derivative of order α of the function $f(t)$ is defined by*

$$\partial_t^\alpha f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s)ds, \tag{2}$$

where $t>0$, $m-1<\alpha\le m$, and $m=1,2,...;$ $\Gamma(z)$ is the Euler's gamma-function.

Definition 2.2. *[8] The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(t)$ is defined as follows*

$$E_{\alpha,\beta}(t)=\sum_{k=0}^{+\infty}\frac{t^k}{\Gamma(\alpha k+\beta)},\quad \alpha,\beta>0,\ \ |t|<\infty.$$

Lemma 2.1. *[7] Consider the following Cauchy problem*

$$\begin{cases}\partial_t^\alpha u(t)-\lambda u(t)=f(t),\ \ t>0,\\ u^{(k)}(t)=uk,\ \ k=0,...,\ m-1,\end{cases} \tag{3}$$

where $\alpha,u_k,\lambda\in R$ such that $m-1<\alpha\le m$, $m\in N$. Then the solution of (3) can be written as follows:

$$u(t)=\sum_{k=0}^{m-1}u_k t^k E_{\alpha,k+1}(\lambda t^\alpha)+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)f(s)ds. \tag{4}$$

We have the following lemma for two-parameter Mittag-Leffler functions [8]:

Lemma 2.2. *If $\alpha<2$, β is any arbitrary real number, μ is such that $\pi\alpha/2<\mu<\min\{\pi,\pi\alpha\}$ and C is a real constant, then*

$$|E_{\alpha,\beta}(z)|\le\frac{C}{1+|z|}\le C,\ \ (\mu\le|\arg(z)|\le\pi),\ \ |z|\ge 0.$$

2.2. Function spaces

Let $t\in[0,T]$ and that, for every t, or at least for a.e. t, the function $u(\cdot,t)$ belongs to a separable Hilbert space V(e.g. $L^2(\Omega)$ or $H^1(\Omega)$) Then, one can consider u as a function of the real variable t with values into V:

$$u:[0,T]\to V.$$

The set $C\left([0,\ T];\ V\ \right)$ of continuous functions $u:[0,T]\to V$, equipped with the norm

$$\|u(\cdot,t)\|_{C([0,T];V)}=\max_{0\le t\le T}\|u(\cdot,t)\|_V \tag{5}$$

forms a Banach space.

3. DEPENDENCE OF THE BOUNDARY CONDITIONS ON THE DEGREE OF DEGENERATION

In this section, we study the dependence of the boundary conditions on the degree of degeneration. For this aim, let us consider the following equation

$$A\upsilon\equiv-\frac{d}{dx}(x^{\beta}\frac{d\upsilon}{dx})=f(x), \tag{6}$$

where $f(x)\in L^2(0,1)$ is a given function, $\beta\in R$ such that $0<\beta<2$ and $\beta\neq 1$.

The domain $D(A)$ of the operator A consists of functions satisfying the following requirements:

$a)$ $\upsilon(x)\in C[0,1]$, the function $x^{\beta}\upsilon'(x)$ has a continuous derivative on $[0,1]$ and $\frac{d}{dx}\left(x^{\beta}\upsilon'(x)\right)\in AC[0,1]$;

$b)$ $A\upsilon\in L^2(0,1)$;

$c)$ satisfies the boundary condition

$$\upsilon(1)=0. \tag{7}$$

Let us determine the conditions that need to be the imposed at $x=0$ for a unique solution to the given problem {(6),(7)} to exist.

For this aim, let us consider the following scalar product

$$\left(-\frac{d}{dx}\left(x^{\beta}\frac{d\upsilon}{dx}\right),\upsilon\right)=-\int_0^1\frac{d}{dx}\left(x^{\beta}\frac{d\upsilon}{dx}\right)\upsilon dx.$$

Hence, applying the rule of integration by parts, we obtain

$$\left(-\frac{d}{dx}\left(x^{\beta}\frac{d\upsilon}{dx}\right),\upsilon\right)=-\upsilon\left(x^{\beta}\frac{d\upsilon}{dx}\right)\Bigg|_0^1+\int_0^1 x^{\beta}\left(\frac{d\upsilon}{dx}\right)^2dx. \tag{8}$$

We will define the conditions under which the first term on the right-hand side of (8) vanishes.

Let $1<\beta<2$. We will show that for these values of β, the following equality

$$\lim_{x\to 0}x^{\beta}\upsilon'(x)=0 \tag{9}$$

is valid.

We assume that converse, i.e. there exists some non-zero constant b such that

$$\lim_{x\to 0}x^{\beta}\upsilon'(x)=b.$$

For instance, let $b>0$. Then for $0<p<b$ and sufficiently small x the following inequality holds:

$$\upsilon'(x)>\frac{p}{x^{\beta}}.$$

Due to the last inequality the following integral

$$\int_0^x \upsilon'(x)dx$$

is divergent, contradicting the continuity of $\upsilon(x)$. From this contradiction, we get the proof of equality (9).

Thus, we proved that if $1<\beta<2$ we do not need any conditions at $x=0$ for vanishing the first term on the right-hand side of (8).

Now, we consider the case $0<\beta<1$. In this case, we have

$$\lim_{x\to 0} x^{\beta}\upsilon'(x)\neq 0.$$

Thus, we proved that for any function $\upsilon\in D(A)$, in case $0<\beta<1$, the condition $\upsilon(0)=0$ is a necessary condition for vanishing the first term on the right-hand side of (8) and in case $\beta>1$, we do not need any conditions at $x=0$.

4. MAIN PROBLEM OUTLINE AND EXPLORATION

For equation (1), we study the following boundary value problem:

Problem 4.1. Find a solution in the domain Ω of the equation (1) satisfying the following initial

$$u(x,0)=\varphi(x),\ \ 0\le x\le 1, \tag{10}$$

and the boundary conditions:

$$\frac{\partial^{\nu}u(0,t)}{\partial x^{\nu}}=0;\ \ \nu=\overline{0,-[\beta]};\ \ u(1,t)=0,\ 0\le t\le T, \tag{11}$$

where φ is a given function, $[\beta]$ denotes the integer part of β.

First, we will examine the form of the boundary condition (11) for different values of β.

Let $\beta\in(0,1)$. Then, the condition (11) takes the form

$$u(0,t)=0,\ \ u(1,t)=0,\ \ 0\le t\le T. \tag{12}$$

For the values of $\beta\in(1,2)$, the condition (11) has the form

$$u(1,t)=0,\ \ \ 0\le t\le T, \tag{13}$$

so we do not need any conditions on the line $x=0$.

Therefore, the condition (11) encompasses the conditions (12), (13).

5. SPECTRAL PROBLEM

For studying problem 4.1 we use the method of separation variables, i.e. we will seek the solution of the equation (1) satisfying the boundary conditions (11) in the form

$$u(x,t)=T(t)\upsilon(x),$$

where $T(t)$ and $\upsilon(x)$ are unknown functions.

Then, we obtain the following spectral problem for $\upsilon(x)$:

$$-(x^{\beta}\upsilon'(x))'=\lambda\upsilon(x), \tag{14}$$

$$\upsilon^{(\mu)}(0)=0,\ \ \mu=\overline{0,-[\beta]},\ \ \upsilon(1)=0. \tag{15}$$

For proving the existence of the eigenvalues and eigenfunctions of the spectral problem {(14),(15)} we consider the following operator

$$A\upsilon\equiv-(x^{\beta}\upsilon')'$$

with the domain $D(A)$ that consists of function with the following properties:

1) $\upsilon\in C[0,1]$;

2) $x^{\beta}\upsilon'(x)\in C[0,1]$ and $(x^{\beta}\upsilon'(x))'\in AC[0,1]$, $A\upsilon\in L^2(0,1)$;

3) $\upsilon^{(\mu)}(0)=0$, $\mu=\overline{0,-[\beta]}$, $\upsilon(1)=0$.

We denote by $\dot{W}^1_{2,\beta}$ the closure of the set $D(A)$ with the following norm

$$\|\upsilon\|^2_{W^1_{2,\beta}}=\int_0^1\left[\upsilon^2(x)+x^{\beta}\upsilon'^2(x)\right]dx.$$

For proving the existence of the eigenfunctions and eigenvalues of the spectral problem {(14),(15)} we use the variational method [9].

To do this we show the positive definiteness of the operator A.

Theorem 5.1. Let $0<\beta<2$ and let $\beta\neq 1$. Then the operator A is positive definite in $L^2(0,1)$.

Proof. First, we will prove that the operator A is a symmetric. Since the domain of the definition of the operator contains finite functions on $(0,1)$, the domain of the operator $D(A)$ is dense in $L^2(0,1)$.

Now, for any functions $\upsilon,\omega\in D(A)$, we consider the following scalar product

$$(A\upsilon,\omega)=\int_0^1(x^{\beta}\upsilon')'\omega dx.$$

Applying the rule of integration by parts, and considering the conditions (15) from the last, we obtain

$$(A\upsilon,\omega)=\int_0^1 x^{\beta}\upsilon'\omega' dx. \tag{16}$$

Hence, by applying integration by parts once again and taking (15) into account, we get

$$(A\upsilon,\omega)=\int_0^1 x^{\beta}\omega'\upsilon' dx=(\upsilon,A\omega). \tag{17}$$

Thus, we obtain the equality $(A\upsilon,\omega)=(\upsilon,A\omega)$, from which it follows that the operator A is symmetric.

Now, we shall show that for any $\upsilon\in D(A)$, there exists a positive constant γ, and the operator A satisfies the following positive definiteness inequality

$$(A\upsilon,\upsilon)\geq\gamma\|\upsilon\|^2_{L^2(0,1)}. \tag{18}$$

From the equality (17), in case $\omega=\upsilon$, we obtain

$$(A\upsilon,\upsilon)=\int_0^1 x^{\beta}\upsilon'^2(x)dx. \tag{19}$$

Since $\upsilon(1)=0$, we can write the following equality

$$\upsilon(x)=-\int_0^1\upsilon'(t)dt.$$

Using this equality, we have

$$\int_0^1 \upsilon^2(x)dx = \int_0^1 \left(\int_x^1 \upsilon'(t)dt \right)^2 dx = \int_0^1 \left(\int_x^1 \frac{1}{\sqrt{t^\beta}} \sqrt{t^\beta}\,\upsilon'(t)dt \right)^2 dx .$$

Hence, applying the Cauchy-Schwarz inequality, we obtain

$$\int_0^1 \upsilon^2(x)dx \le \int_0^1 \left(\int_x^1 t^{-\beta}dt \right)\left(\int_x^1 t^\beta \left[\upsilon'(t)\right]^2 dt \right) \le$$

$$\le \int_0^1 \left(\int_x^1 t^{-\beta}dt \right)dx \left(\int_0^1 t^\beta \left[\upsilon'(t)\right]^2 dt \right). \tag{20}$$

From the last, we have

$$\int_0^1 \upsilon^2(x)dx \le \int_0^1 x^\beta \upsilon'^2(x)dx \int_0^1 \int_x^1 t^{-\beta}dtdx.$$

Since $\beta \neq 1$, we have

$$\int_0^1 \int_x^1 t^{-\beta}dtdx = \frac{1}{1-\beta} - \frac{1}{(1-\beta)(2-\beta)} = \frac{1}{2-\beta} > 0.$$

Then, introducing

$$\gamma = 2 - \beta$$

we come the proof of the inequality (18).

Thus, the proof of Theorem 5.1 is complete.

Now, we consider the Friedrichs extension of the operator A, and denote this extension by the same notation, A.

Theorem 5.2. Let $0 < \beta < 2$ and $\beta \neq 1$. Then, the operator A has a discrete spectrum.

Proof. Since, the operator A is positive definite, we introduce the energetic space H_A of the operator A with the following norm

$$\|\upsilon\|_{H_A}^2 = \int_0^1 x^\beta \upsilon'^2(x)dx.$$

It is easy to show that H_A, as a set of functions, coincides with the space $\dot{W}^1_{2,\beta}(0,1)$.

Moreover, the norms of these two spaces are equivalent.

Let M be a set of functions υ for which the norm in the energetic space H_A is bounded, i.e. $\|\upsilon\|_{H_A} \le c$, where c is a finite constant number.

Then, from (17) and (18) it follows that

$$\|\upsilon\|_{W^1_{2,\beta}}^2 \equiv \int_0^1 \left[\upsilon^2(x) + x^\beta \upsilon'^2(x) \right] dx \le const .$$

Then, by the Kondrashov embedding theorems for weighted classes [10], the set M is compact in the space into which it is embedded, specifically:

a) in the space of continuous functions if $0 < \beta < 1$;

b) in the space L^q if $\beta > 1$, where $q \in R$ such that $q < 2/(\beta - 1)$.

These statements show that the set M is compact in the space $L^2(0,1)$. Then, based on Theorem 3 (§40) in [9], we conclude that the spectrum of the operator A is discrete, i.e.

the system of eigenfunctions of the operator A is a complete orthonormal system in $L^2(0,1)$ and orthogonal in H_A:

$$\int_0^1 \upsilon_i(x)\upsilon_j(x)dx = \begin{cases} 1, & i=j; \\ 0, & i\neq j; \end{cases} \tag{21}$$

$$\int_0^1 x^{\beta}\upsilon_i'(x)\upsilon_j'(x)dx = \begin{cases} \lambda_i, & i=j; \\ 0, & i\neq j; \end{cases} \tag{22}$$

where $\{\upsilon_n\}_{n=1}^{+\infty}$ and $\{\lambda_n\}_{n=1}^{+\infty}$ are the eigenfunctions and eigenvalues of the spectral problem {(21),(22)}, respectively.

6. THE ORDER OF FOURIER COEFFICIENTS

In this section, we establish the convergence of some series that will be used throughout the paper.

Let $\{\upsilon_n\}_{n=1}^{+\infty}$ and $\{\lambda_n\}_{n=1}^{+\infty}$ be the eigenfunctions and eigenvalues of the spectral problem {(14),(15)}.

Lemma 6.1. Let $f \in \dot{W}_{2,\beta}^1(0,1)$. Then, the following inequality holds:

$$\sum_{n=1}^{+\infty} \lambda_n f_n^2 \leq \int_0^1 x^{\beta}[f'(x)]^2 dx \tag{23}$$

from which follows the convergence of the series on the left-hand side of (23), where $f_n = \int_0^1 f(x)\upsilon_n(x)dx.$

Proof. By the definition of generalized eigenfunctions for any $f \in \dot{W}_{2,\beta}^1(0,1)$ the following equality holds

$$\int_0^1 x^{\beta}\upsilon_n'(x)f'(x)dx = \lambda_n f_n . \tag{24}$$

Let us consider the following non-negative expression:

$$\int_0^1 x^{\beta}\left(f'(x) - \sum_{i=1}^{n} f_i\upsilon_i'(x)\right)^2 dx \geq 0.$$

From the last, we obtain

$$\int_0^1 x^{\beta} f'^2(x)dx - 2\sum_{i=1}^{n} f_n \int_0^1 x^{\beta} f'(x)\upsilon_n'(x)dx +$$

$$+\sum_{i=1}^{n} f_n^2 \int_0^1 x^{\beta}\upsilon_n'(x)dx + 2\sum_{\substack{i,j=1 \\ i\neq j}}^{n} f_i f_j \int_0^1 x^{\beta}\upsilon_i'(x)\upsilon_j'(x)dx \geq 0.$$

Considering the last equality and (22), (24), we have

$$\sum_{i=1}^{n} \lambda_i f_i^2 \leq \int_0^1 x^{\beta}[f'(x)]^2 dx.$$

Hence, passing to the limit as $n \to +\infty$ we come to the inequality (23).

Let us introduce the following notation

$$A^m f = \underbrace{A(A(A...(Af)...))}_{m-times}.$$

Lemma 6.2. Let the following conditions be satisfied

(1) for all even *m* numbers $f, Af, \dots, A^{\frac{m}{2}} f \in \dot{W}^1_{2,\beta}(0,1);$

(2) for all odd *m* numbers $f, Af, \dots, A^{\frac{m-1}{2}} f \in \dot{W}^1_{2,\beta}(0,1),\ A^{\frac{m+1}{2}} f \in L^2(0,1)$.

Then, the following Bessel-type inequalities hold true:

$$\sum_{n=1}^{+\infty} \lambda_n^{m+1} f_n^2 \le \begin{cases} \int_0^1 [(A^{\frac{m}{2}} f)']^2 dx, & for\, even\, m, \\ \int_0^1 [A^{\frac{m+1}{2}} f]^2 dx, & for\, odd\, m. \end{cases} \tag{25}$$

Proof. Assume that *m* is even natural number. Then, applying the rule of integration by parts twice, from (24) we have

$$(Af, \upsilon_n) = \lambda_n f_n = \lambda_n (f, \upsilon_n). \tag{26}$$

Replacing f by Af in (26), we obtain

$$(A^2 f, \upsilon_n) = \lambda_n (Af, \upsilon_n) = \lambda_n^2 f_n.$$

Similarly, one can show that the following equality is valid:

$$(A^{\frac{m}{2}} f, \upsilon_n) = \lambda_n^{\frac{m}{2}} f_n. \tag{27}$$

Equation (27) shows that $\lambda_n^{\frac{m}{2}} f_n$ is the Fourier coefficient of the function $A^{\frac{m}{2}} f$ for the system of the eigenfunctions of the spectral problem {(14) (15)}.

Since, $A^{\frac{m}{2}} f \in \dot{W}^1_{2,\alpha}(0,1)$ by applying Lemma 6.2, we have

$$\sum_{n=1}^{+\infty} \lambda_n \lambda_n^m f_n^2 = \sum_{n=1}^{+\infty} \lambda_n^{m+1} f_n^2 \le \int_0^1 x^\beta [(A^{\frac{m}{2}} f)']^2 dx.$$

Thus, we have proved Lemma 6.2 for even *m*.

Now, we consider the case where *m* is odd.

In this case, from (27), we have

$$\left(A^{\frac{m+1}{2}} f, \upsilon_n \right) = \lambda_n^{\frac{m+1}{2}} f_n.$$

Under the assumptions of Lemma 6.2, we have $A^{\frac{m+1}{2}} f \in L^2(0,1)$. Then, applying Bessel's inequality, we obtain

$$\sum_{n=1}^{+\infty} \left(A^{m+1} f, \upsilon_n \right)^2 = \sum_{n=1}^{+\infty} \lambda_n^{m-1} f_n^2 \le \int_0^1 \left[A^{\frac{m+1}{2}} f \right]^2 dx.$$

The proof of Lemma 6.2 is complete.

Lemma 6.3 Let $f \in C\left([0,T], \dot{W}^1_{2,\beta}(0,1)\right),\ Af \in C\left([0,T], L^2(0,1)\right)$. Then the following series

$$\sum_{n=1}^{+\infty} \lambda_n^2 f_n^2(t)$$

converges absolutely and uniformly in $\left[0,T\right]$.

Proof. By employing Parseval's equality for the function $Af(x,t),$ we have

$$\sum_{n=1}^{+\infty}\lambda_n^2 f_n^2(t)=\int_0^1\left[Af(x,t)\right]^2 dx.$$

Since $Af(x,t)\in C([0,T],L^2(0,1))$ based on Dini's theorem, we have the proof of Lemma 6.3.

Lemma 6.4. Let $f(x,t),\ Af(x,t)\in C([0,T],\dot{W}^1_{2,\beta}(0,1)).$ Then the following series

$$\sum_{n=1}^{+\infty}\lambda_n^3 f_n^2(t)$$

converges absolutely and uniformly in $\left[0,T\right]$.

Proof. Since $(Af,\upsilon_n)=\lambda_n f_n(t)$ and $Af(x,t)\in C([0,T],\dot{W}^1_{2,\beta}(0,1)),$ by applying Lemma 6.1, we have

$$\sum_{n=1}^{+\infty}\lambda_n^3 f_n^2(t)\le\int_0^1 x^{\beta}[(Af)']^2 dx<+\infty.$$

Since $Af(x,t)\in C([0,T],\dot{W}^2_{1,\beta}(0,1))$ based on Dini's theorem, we have the proof of Lemma 6.4.

7. EXISTENCE OF THE SOLUTION TO THE PROBLEM.

In this section, we will prove the existence and the uniqueness of the solution to the Problem 4.1.

We consider two cases.

Case 1. $0<\beta<1$.

Case 2. $1<\beta<2$.

First, we consider Case 1.

Let $\{\upsilon_k\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$ be the eigenfunctions and eigenvalues of the spectral problem {(14),(15)}. We will seek a solution to the problem in the following form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\upsilon_k(x), \tag{28}$$

where $u_k(t)=(u,\upsilon_k)_{L^2(0,1)}$ for $k\in\mathrm{N}$.

Substituting (28) into equation (1) and introducing $(f,\upsilon_k)_{L^2(0,1)}=f_k(t),$ from (1), we obtain

$$\partial_t^{\alpha}u_k(t)+\lambda_k u_k(t)=f_k(t),\ k\in\mathrm{N},\ t\in(0,T). \tag{29}$$

Moreover, from the initial condition (10), we have

$$u_k(0)=\varphi_k,\ k\in\mathrm{N}, \tag{30}$$

where $\varphi_k=(\varphi,\upsilon_k)_{L^2(0,1)}$.

Using Lemma 2.1, it is easy to see that the solution to the problem {(29),(30)} has the form

$$u_k(t)=\varphi_k E_{\alpha,1}(-\lambda_k t^\alpha)+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}[-\lambda_k(t-s)^\alpha]f_k(s)ds. \tag{31}$$

Substituting (31) into (28), we obtain

$$u(x,t)=\sum_{k=1}^{+\infty}\Big\{\varphi_k E_{\alpha,1}\left(-\lambda_k t^\alpha\right)+$$
$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left[-\lambda_k(t-s)^\alpha\right]f_k(s)ds\Big\}\upsilon_k(x). \tag{32}$$

Applying the rule of integrating by parts, we rewrite (32) in the form

$$u(x,t)=\sum_{k=1}^{+\infty}\{\varphi_k\, E_{\alpha,1}\left(-\lambda_k t^\alpha\right)+E_{\alpha,\alpha+1}\left(-\lambda_k t^\alpha\right)f_k(0)+$$
$$+\int_0^t (t-s)^{\alpha}E_{\alpha,\alpha+1}\left[-\lambda_k(t-z)^\alpha\right]f_k'(z)dz\Big\}\upsilon_k(x). \tag{33}$$

Theorem 7.1. Let the following conditions be satisfied

1) $0<\beta<1$;
2) $\varphi,\ A\varphi\in \dot W^1_{2,\beta}(0,1)$
3) $f,\ Af\in \dot W^1_{2,\beta}(0,1)$ with respect to x;
4) $\dfrac{\partial f}{\partial t}\in C\left(\left[0,T\right],\ \dot W^1_{2,\beta}(0,1)\right).$

Then, the function $u(x,t)$ defined by (33) will be the classical solution to problem.

Proof. To prove Theorem 7.1, we will show the continuity of the functions $\partial_t^\alpha u$ and $\dfrac{\partial}{\partial x}\left[x^\beta\dfrac{\partial u}{\partial x}\right]$ in $\bar\Omega$. Formally differentiating (33) and considering (14), we obtain

$$\frac{\partial}{\partial x}\left(x^\beta\frac{\partial u}{\partial x}\right)=\sum_{k=1}^{+\infty}\lambda_k u_k(t)\upsilon_k(x),$$

or we can write the last, in the following form

$$Au=\sum_{k=1}^{+\infty}\lambda_k\Big\{\varphi_k E_{\alpha,1}\left(-\lambda_k t^\alpha\right)+E_{\alpha,\alpha+1}\left(-\lambda_k t^\alpha\right)f_k(0)+$$
$$+\int_0^t (t-s)^{\alpha}E_{\alpha,\alpha+1}\left(-\lambda_k(t-z)^\alpha\right)f_k'(z)dz\Big\}\upsilon_k(x). \tag{34}$$

From (34), we obtain

$$\|Au\|^2_{H_A}=\sum_{k=1}^{+\infty}\lambda_k^3\Big\{\varphi_k E_{\alpha,1}\left(-\lambda_k t^\alpha\right)+E_{\alpha,\alpha+1}\left(-\lambda_k t^\alpha\right)f_k(0)+$$
$$+\int_0^t (t-s)^{\alpha}E_{\alpha,\alpha+1}\left[-\lambda_k(t-z)^\alpha\right]f_k'(z)dz\Big\}^2 \tag{35}$$

Hence, applying $(a+b+c)^2\le 3\left(a^2+b^2+c^2\right)$, we have

$$\|Au\|_{H_A}^2 \le 3\sum_{k=1}^{+\infty}\lambda_k^3\varphi_k^2E_{\alpha,1}^2\left(-\lambda_k t^\alpha\right)+3\sum_{k=1}^{+\infty}\lambda_k^3 f_k^2(0)E_{\alpha,\alpha+1}^2\left(-\lambda_k t^\alpha\right)$$

$$+3\sum_{k=1}^{+\infty}\lambda_k^3\left(\int_0^t (t-s)^\alpha E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^\alpha\right] f_k'(z)dz\right)^2.$$

Applying Lemma 2.2 and Cauchy-Schwarz inequality, we have

$$\lambda_k^3\left(\int_0^t (t-s)^\alpha E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^\alpha\right] f_k'(z)dz\right)^2 \le$$

$$\le \lambda_k^3\left(\int_0^t (t-s)^\alpha \frac{C}{1+\lambda_k (t-z)^\alpha}\left|f_k'(z)\right|dz\right)^2 \le$$

$$\le C^2\lambda_k\left(\int_0^t \left|f_k'(z)\right|dz\right)^2 \le C^2\lambda_k\int_0^t dz\int_0^t f_k'^2(z)dz \le$$

$$\le C^2\lambda_k T\int_0^T f_k'^2(z)dz.$$

Considering the last inequality and boundedness of the Mittag-Leffler function, we obtain

$$\|Au\|_{H_A}^2 \le C_1\sum_{k=1}^{+\infty}\lambda_k^3\varphi_k^2 + C_2\sum_{k=1}^{+\infty}\lambda_k^3 f_k^2(0) + C_3\sum_{k=1}^{+\infty}\lambda_k\int_0^T f_k'^2(s)ds, \tag{36}$$

where C_1, C_2 and C_3 are the constants not depend on k

The convergence of the two series on the right-hand side of (36) follows from Lemma 6.2, and the convergence of the third series follows from Lemma 6.4, with the subsequent application of the theorem on term-by-term integration of uniformly converging series.

Consequently, we have obtained that

$$\|Au\|_{H_A}^2 \le const$$

from which by Kondrashov embedding theorem for weighted classes [10], we have $Au \in C[0,1]$. The continuity of Au with respect to t follows from the uniform convergence of the series

$$\sum_{k=1}^{+\infty}\lambda_k^3 u_k^2(t)$$

in $[0,T]$. Thus, $Au \in C(\overline{\Omega})$.

Theorem 7.2. Let the following conditions be satisfied:

1) $1<\beta<2$
2) $\varphi \in \dot{W}_{2,\beta}^1(0,1),\ A\varphi \in L^2(0,1);$
3) $f \in C\left([0,T];\dot{W}_{2,\beta}^1(0,1)\right),\ Af \in C\left([0,T],L^2(0,1)\right)$
4) $\dfrac{\partial f}{\partial t} \in C\left([0,T],\dot{W}_{2,\beta}^1(0,1)\right).$

Then, the function $u(x,t)$ defined by (32) will be a function such that $Au \in C\left([0,T],\, L^2(0,1)\right)$ and $\partial_t^\alpha u \in C\left([0,T],\, L^2(0,1)\right)$.

Proof. Using (32), we have

$$\|A\|_{L^2(0,1)}^2 = \sum_{k=1}^{+\infty} \lambda_k^2 u_k^2(t).$$

It is easy to verify that the following estimate holds true:

$$\|A\|_{L^2(0,1)}^2 \le C_4 \sum_{k=1}^{+\infty} \lambda_k^2 \varphi_k^2 + C_5 \sum_{k=1}^{+\infty} \lambda_k^2 f_k^2(0) + C_6 \sum_{k=1}^{+\infty} \lambda_k^2 \int_0^T f_k'^2(s)\,ds,$$

where C_4, C_5 and C_6 are constants not depend on k .

The convergence of the two series follows from the conditions Theorem 7.2 and Lemmas 6.2 and 6.3. Additionally, the convergence of the third series follows from Lemma 6.3.

The uniqueness of the solution can be proved using the completeness property of the system of eigenfunctions $\{\upsilon_k(x)\}\big|_{k=1}^{+\infty}$ in $L^2(0,1)$.

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