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INVERSE SOURCE PROBLEM FOR A DEGENERATE SUBDIFFUSION EQUATION

BUZILADIGAN SUBDIFFUZIYA TENGLAMASI UCHUN MANBANI ANIQLASH HAQIDAGI TESKARI MASALA

ОБРАТНАЯ ЗАДАЧА ИСТОЧНИКА ДЛЯ ВЫРОЖДЕННОГО УРАВНЕНИЯ СУБДИФФУЗИИ

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Abstract

In the present paper, the inverse source problem is formulated and studied for a degenerate sub-diffusion equation in a rectangular domain. By employing the method of separation of variables, a spectral problem was obtained for an ordinary differential equation concerning the spatial variable. The existence of the eigenvalues and eigenfunctions of this spectral problem was established by equivalently reducing it to a homogeneous second-kind Fredholm integral equation with symmetric kernels. Using the theory of integral equations, the existence of the eigenvalues and eigenfunctions was further confirmed. The solution to the inverse source problem is expressed as the sum of a Fourier series over the system of eigenfunctions of the spectral problem. The uniform convergence of the obtained series of solution were proved.

Annotatsiya

Ushbu maqolada to'g'ri to'rtburchak soha chegarasida buziladigan subdiffuziya tenglamasi uchun faqat fazo o'zgaruvchisiga bog'liq manbani aniqlash haqidagi teskari masala bayon qilingan va o'rganilgan. O'zgaruvchilarni ajratish usulidan foydalanib, oddiy differensial tenglama uchun spektral masala hosil qilingan. Spektral masalaning xos sonlari va xos funksiyalarining mavjudligi uni simmetrik yadroli bir jinsli 2-tur Fredgolm integral tenglamasi ekvivalent keltirish orqali integral tenglamalar nazariyasidan foydalanib isbotlangan. Teskari masalaning yechimi, spektral maslaning xos funksiyalar sistemasi orqali Fure qatorlari ko'rinishida topilgan. Hosil qilingan qatorlarning yaqinlashuvchiligi isbotlangan.

Аннотация

В данной работе сформулирована и исследована обратная задача определения источника для вырожденного субдиффузионного уравнения в прямоугольной области. С использованием метода разделения переменных была получена спектральная задача для обыкновенного дифференциального уравнения относительно пространственной переменной. Существование собственных значений и собственных функций данной спектральной задачи было установлено путем эквивалентного сведения к однородному интегральному уравнению Фредгольма второго рода с симметрическим ядром. Далее, с применением теории интегральных уравнений, существование собственных значений и собственных функций было дополнительно подтверждено. Решение обратной задачи представлено в виде суммы ряда Фурье по системе собственных функций спектральной задачи. Доказана равномерная сходимость полученного ряда решений.

Key words: *time-fractional diffusion equation, degenerate equation, inverse problem, spectral method.*

Kalit so'zlar: *vaqt bo'yicha kasr tartibli diffuziya tenglamasi, buziladigan differensial tenglama, teskari masala, spektral metod.*

Ключевые слова: *уравнение дробно-временной диффузии, вырожденное уравнение, обратная задача, спектральный метод.*

INTRODUCTION

Fractional calculus is an area of mathematical analysis that deals with the study and application of integrals and derivatives of arbitrary order. In recent decades, fractional calculus has gained increasing significance due to its applications in various fields of science, engineering, and the theory of complex systems. For more information on this research, we refer the reader to [1], [2], [3], [4], and the references therein. During the past few decades, researchers have paid attention to studying the fractional analogs of integer-order differential equations. It can be explained that, on the one hand, various types of real-life processes are represented by such equations, and on the other hand, it is an intrinsic necessity of the theory of differential equations.

LITERATURE REVIEW

Along with initial-boundary value problems, researchers have also focused on studying inverse problems and problems with non-local conditions for one-term and multi-term time-fractional diffusion equations. In [5], M.Ruzhansky et al. considered non-local problems for the time-fractional diffusion-wave equation with general operators having a discrete spectrum. They proved the existence and uniqueness of the solutions to the considered problems. The work [6] by E. Karimov et al. is devoted to the study of a non-local initial problem for a multi-term time-fractional space-singular equation. In this direction, we also note the work [7] on the inverse source problem for the time-fractional space-degenerate heat equation and [8] on inverse problems for a multi-parameter space-time fractional diffusion equation with non-local conditions. Additionally, the work [9] is devoted to studying the inverse problem of restoring the initial data of a solution, classical in time, and with values in the space of periodic spatial distributions for a time-fractional diffusion equation and a diffusion-wave equation. It should be noted that the aforementioned works consider only non-degenerate time-fractional equations.

However, both local and non-local boundary value problems for degenerate partial differential equations involving fractional derivatives of the unknown function remain poorly explored. The study of boundary value problems for such equations is of great significance not only from a theoretical perspective but also from a practical one, as these equations and their corresponding problems arise in the mathematical modeling of various phenomena in gas and hydrodynamics, the theory of small surface deflections, mathematical biology, and other scientific fields.

3. THE MAIN RESULTS

3.1. Formulation of the problem.

In the rectangular domain $\Omega=\{(x,t):0<x<1,0<t<T\}$, we consider the following equation

$$ {}_cD_{0t}^{\alpha}u(x,t)=(x^{\beta}u_x)_x+f(x) \tag{1}$$

where $u(x,t)$, $f(x)$ are the unknown functions, and ${}_CD_{0t}^{\alpha}$ is Caputo fractional differential operator α order [10]

$${}_CD_{0t}^{\gamma}g(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{g^{(n)}(z)dz}{(t-z)^{\gamma-n+1}}, \qquad (n=\mathrm{Re}(\gamma)+1,t>0)$$

α,β are given real numbers, such that $0<\alpha<1$, $0<\beta<1$.

Assuming $f(x)$ is a known function, we will first give the definition of the regular solution of equation (1).

Definition. *A function $u(x,t)$ satisfying equation (1) in the domain Ω, and the following conditions $u(x,t)\in C(\overline{\Omega})$, ${}_CD_{0t}^{\alpha}u(x,t)$, $(x^{\beta}u_x)_x\in C(\Omega)$ is called regular in the domain Ω solution of the equation (1).*

For equation (1) we investigate the following inverse-source problem:

Problem. *Show the existence and uniqueness of the pair of functions $\{u,f\}$ with the following properties:*

1. *$u(x,t)$ is the regular solution of the equation (1);*
2. *$u(x,t)$ satisfies the following initial*

$$u(x,0)=\varphi(x), \qquad 0\le x\le 1, \tag{2}$$

and the following nonlocal boundary conditions:

$$pu(0,t)=qu(1,t),\ q\left[x^{\beta}u_x\right]\Big|_{x=0}=p\left[x^{\beta}u_x\right]\Big|_{x=1},\quad 0\le t\le T, \tag{3}$$

and the following over-determination condition:

$$u(x,T)=\psi(x),\qquad 0\le x\le 1. \tag{4}$$

where $\varphi(x)$ *is a given function such that* $p\varphi(0)=q\psi(1)$, $q\left[x^{\beta}\varphi'(x)\right]\Big|_{x=0}=p\left[x^{\beta}\varphi'(x)\right]\Big|_{x=1}$ *and* p,q *are given real numbers such that* $p^2+q^2\ne 0$ *.*

3.2. Spectral problem.

We will seek a solution of the homogeneous equation corresponding equation (1) satisfying the conditions (3) in the form:

$$u(x,t)=T(t)X(x).$$

Then, concerning $X(x)$, we will have the following spectral problem:

$$LX\equiv-\left[x^{\beta}X'(x)\right]'=\lambda X(x),\quad 0<x<1, \tag{5}$$

$$pX(0)=qX(1),\ \ q\left[x^{\beta}X'(x)\right]\Big|_{x=0}=p\left[x^{\beta}X'(x)\right]\Big|_{x=1}. \tag{6}$$

We will find the eigenvalues and eigenfunction of the problem {(5),(6)}. First, we will determine the sign of the eigenvalues, for this aim, by multiplying both sides of equation (5) to the function $X(x)$ and integrating with respect x over the segment $[0,1]$, and then integrating by parts to the integral on the left-hand side of the obtained equality and using the condition (6), we get the following equation

$$\int_0^1 x^{\beta}\left[X'(x)\right]^2dx=\lambda\int_0^1 X^2(x)dx.$$

From the last equation, it follows that $\lambda\ge 0$. Let $\lambda=0$, then from the (5), we have

$$\left[x^{\beta}X'(x)\right]'=0.$$

It is easy to see that the general solution of the last equation has the following form

$$X(x)=C_1\frac{x^{1-\beta}}{1-\beta}+C_2,$$

where C_1, C_2 are arbitrary constants.

From this, due to the condition $pX(0)=qX(1)$, $p\ne q$, we obtain $X(x)\equiv 0$, $0\le x\le 1$. Consequently, the spectral problem {(5),(6)} has nontrivial solutions only when λ >0, provided that $p\ne q$.

To establish the existence of the eigenvalues and the eigenfunctions of the spectral problem {(5), (6)}, we'll use the method of Green functions. For this aim, we will construct the Green's function $G(x,s)$. It satisfies the following properties:

1. $G(x,s)$ is a continuous function for all $x,s\in[0,1]$;
2. In each of the intervals $(0,s)$ and $(s,1)$, the function $\partial G(x,s)/\partial x$ is continuous, but at $x=s$ it has a jump $-\frac{1}{s^{\beta}}$;
3. In the intervals $(0,s)$ and $(s,1)$, the function $G(x,s)$, considered as function of the variable x, satisfies the homogeneous equation $LG(x,s)=0$;

4. For $\forall s \in (0,1)$, it satisfies the boundary conditions (6) concerning the variable x.

It can be demonstrated that a function fulfilling the aforementioned criteria exists uniquely and is defined by the following expression

$$G(x,s)=\begin{cases} \dfrac{p^2x^{1-\beta}-pqx^{1-\beta}+q^2+pqs^{1-\beta}-q^2s^{1-\beta}}{(1-\beta)(p-q)^2}, & 0\le x\le s; \\ \dfrac{p^2s^{1-\beta}-pqs^{1-\beta}+q^2+pqx^{1-\beta}-q^2x^{1-\beta}}{(1-\beta)(p-q)^2}, & s\le x\le 1. \end{cases} \tag{7}$$

Then the problem {(5),(6)} is equivalent to the following homogeneous integral equation with the symmetric kernel $G(x,s)$

$$X(x)=\lambda\int_0^1 G(x,s)X(s)ds. \tag{8}$$

Since the kernel $G(x,s)$ is continuous, symmetric, and positive $(\lambda>0)$, the integral equation, and hence the spectral problem {(5),(6)}, has a countable set of eigenvalues:

$$0<\lambda_1<\lambda_2<\lambda_3<...<\lambda_n...,\quad \lambda_n\to+\infty,$$

and the corresponding eigenfunctions $X_1(x),X_2(x),...,X_n(x),...$ form an orthonormal system in $L_2(0,1)$. Moreover, functions that can be expressed by the kernel $G(x,s)$ can be represented as Fourier series by the system of eigenfunctions [10].

Let $\{X_n(x)\}\big|_{n=1}^{+\infty}$ and $\{\lambda_n\}\big|_{n=1}^{+\infty}$ be eigenfunctions and eigenvalues of the spectral problem {(5),(6)} respectively.

Lemma 1. *If the function $\omega(x)$ satisfies the conditions: $\omega(x)$, $x^\beta\omega'(x)\in C[0,1]$, $L\omega(x)\in L_2(0,1)$,*

$$p\omega(0)=q\omega(1),$$
$$qx^\beta\omega'(x)\big|_{x=0}=px^\beta\omega'(x)\big|_{x=1}.$$

Then the function $\omega(x)$ can be expanded into an absolutely and uniformly convergent series in terms of the system of eigenfunctions $\{X_n(x)\}\big|_{n=1}^{+\infty}$ on $[0,1]$.

Proof. First, we will show the validity of the following equality under the assumptions of Lemma 1:

$$\omega(x)=\int_0^1 G(x,s)L\omega(s)=\int_0^1 G(x,s)\left[s^\beta\omega'(s)\right]'ds.$$

Indeed, considering properties of the functions $\omega(x)$ and $G(x,s)$, we have

$$\int_0^1 G(x,s)L\omega(s)=\int_0^1 G(x,s)\left[s^\beta\omega'(s)\right]'ds=$$

$$=\left[s^\beta\omega'(s)\right]G(x,s)\Big|_{s=0}^{s=1}-\left[s^\beta\omega'(s)\right]G_s(x,s)\Big|_{s=0}^{s=1}+\int_0^1\omega(s)\left[s^\beta G_s(x,s)\right]_s ds=\omega(x).$$

Consequently, $\omega(x)$ is a function that can be representable through the kernel $G(x,s)$. Moreover, one can easily show that the validity of the following inequality

$$\int_0^1 G^2(x,s)ds \le C_3 = const < +\infty .$$

Then, by the Hilbert-Schmidt theorem [10], the function $\omega(x)$ on the interval $[0,1]$ is expanded into a uniformly convergent Fourier series in terms of the system of eigenfunctions $\{X_n(x)\}\Big|_{n=1}^{+\infty}$.

Lemma 1 has been proved.

3.3. Convergence of series

In this subsection, we will prove the convergence of some series which will be used the prove the existence of the solution to the main problem.

Lemma 2. *The following series converges uniformly on the segment* $[0,1]$*:*

$$\sum_{n=1}^{+\infty}\frac{X_n^{\,2}(x)}{\lambda_n},\ \sum_{n=1}^{+\infty}\frac{X_n'^2(x)}{\lambda_n},\ \sum_{n=1}^{+\infty}\frac{\left\{\left[x^{\alpha}X_n{}'(x)\right]'\right\}^2}{\lambda_n^2}. \tag{9}$$

Proof. Since the kernel $G(x,s)$ of the integral equation (7) is symmetric, positive, and continuous with respect to (x,s), by Mercer's theorem [10], this kernel is represented by an absolutely and uniformly convergent bilinear series:

$$G(x,s)=\sum_{n=1}^{\infty}\frac{X_n(x)X_n(s)}{\lambda_n}.$$

Hence, particularly, when $x=s$, it follows that

$$\sum_{n=1}^{\infty}\frac{X_n^{\,2}(x)}{\lambda_n}=G(x,x)\le C_3=const .$$

Thus, the first series in (9) converges uniformly on the interval [0,1].

Now, we will show the convergence of the second series in (9). Due to (8) and (5), we have

$$X_n'(x)=\lambda_n\int_0^1 G_x(x,s)X_n(s)ds=-\int_0^1 G_x(x,s)\left[s^{\beta}X_n'(s)\right]'ds .$$

Using the rule of integration by parts and considering the boundary condition (6), we obtain

$$X_n'(x)=\int_0^1\left[s^{\beta}X_n'(s)\right]G''_{xs}(x,s)ds .$$

We rewrite the last equality, in the following form

$$\frac{X_n'(x)}{\sqrt{\lambda_n}}=\int_0^1 s^{\frac{\beta}{2}}G''_{xs}(x,s)\left\{\frac{s^{\frac{\beta}{2}}X_n'(s)}{\sqrt{\lambda_n}}\right\}ds . \tag{10}$$

Now, we will show that the system $\left\{s^{\frac{\beta}{2}}X_n'(s)/\sqrt{\lambda_n}\right\}\Big|_{n=1}^{+\infty}$ forms an orthonormal system in $L_2(0,1)$.

By applying the rule of integration by parts and considering (5) and (6), we have

$$\frac{1}{\sqrt{\lambda_n\lambda_m}}\int_0^1 s^{\beta}X_n'(s)X_m'(s)ds=\frac{1}{\sqrt{\lambda_n\lambda_m}}\left\{s^{\beta}X_n'(s)X_m(s)\Big|_{s=0}^{s=1}-\int_0^1\left[s^{\beta}X_n'(s)\right]'X_m(s)ds\right\}=$$

$$=\sqrt{\frac{\lambda_n}{\lambda_m}}\int_0^1 X_n(s)X_m(s)ds=\begin{cases}1, & n=m,\\ 0, & n\neq m.\end{cases} \tag{11}$$

Thus, the system is $\left\{s^{\frac{\beta}{2}}X_n'(s)/\sqrt{\lambda_n}\right\}\Big|_{n=1}^{+\infty}$ an orthonormal system. Therefore, from (10), it follows that $X_n'(x)\big/\sqrt{\lambda_n}$ is the Fourier coefficient of the function $s^{\frac{\beta}{2}}G_s'(x,s)$ concerning the system $\left\{s^{\beta}X_n'(s)\big/\sqrt{\lambda_n}\right\}\Big|_{n=1}^{+\infty}$.

Hence, according to Bessel's inequality [10], we have

$$\sum_{n=1}^{+\infty}\frac{X_n'(x)}{\sqrt{\lambda_n}}\leq\int_0^1 s^{\beta}\left[G_{xs}''(x,s)\right]^2ds. \tag{12}$$

Using the representation (7) of $G(x,s)$, it is easy to verify that the integral in (12) is uniformly bounded for all $x\in[0,1]$. Therefore, the series in (12), the second series in (9), converge uniformly. The convergence of the remaining series is proved similarly.

Lemma 2 has been proved.

Lemma 3. *Let the following conditions be fulfilled:*

$\omega(x),\ x^{\beta}\omega'(x),\ \left[x^{\beta}\omega'(x)\right]'\in C[0,1],\ \mathrm{L}\omega(x)\in L_2(0,1)$; *and* $p\omega(0)=q\omega(1)$.

Then the following inequality holds

$$\sum_{n=1}^{+\infty}\lambda_n\omega_n^2\leq\int_0^1 x^{\beta}\left[\omega'(x)\right]^2dx. \tag{13}$$

Proof. Using the equation (5), we can write

$$\lambda_n^{1/2}\omega_n=\lambda_n^{1/2}\int_0^1\omega(x)X_n(x)dx=\lambda_n^{-1/2}\int_0^1\omega(x)\left[x^{\beta}X_n'(x)\right]'dx.$$

Hence, applying the rule of integration by parts twice and considering the properties of the function $\omega(x)$ and $X_n(x)$, we obtain

$$\lambda_n^{1/2}\omega_n=\int_0^1\left\{x^{\beta/2}\omega'(x)\right\}\left\{\lambda_n^{-1/2}x^{\beta/2}X_n(x)\right\}dx.$$

Thus, the number $\lambda_n^{1/2}\omega_n$ is the Fourier coefficient of the function $x^{\beta/2}\omega'(x)$ concerning the orthonormal system of functions $\left\{\lambda_n^{-1/2}x^{\beta/2}X_n'(x)\right\}\Big|_{n=1}^{+\infty}$. Then, according to Bessel's inequality, the inequality (13).

Lemma 3 has been proved.

Lemma 4. *If the conditions* $\omega(x)$, $x^{\beta}\omega'(x)$, $\left[x^{\beta}\omega'(x)\right]'$, $\mathrm{L}\omega(x)\in C[0,1]$, $x^{\beta/2}\mathrm{L}\omega(x)\in L_2(0,1)$ *hold, and if* $p\omega(0) = q\omega(1)$, $qx^{\beta}\omega'(x)\big|_{x=0} = px^{\beta}\omega'(x)\big|_{x=1}$, *then the following inequality holds:*

$$\sum_{n=1}^{+\infty}\lambda_n^3\omega_n^2 \le \int_0^1 x^{\beta}\left\{\left[L\omega(x)\right]'\right\}^2 dx. \qquad (13)$$

Proof. By equation (5), the following equality holds

$$\lambda_n^{3/2}\omega_n = \lambda_n^{3/2}\int_0^1 \omega(x)X_n(x)dx = \lambda_n^{1/2}\int_0^1 \omega(x)LX_n(x)dx =$$

$$= \lambda_n^{1/2}\int_0^1 \omega(x)\left[x^{\beta}X_n'(x)\right]' dx.$$

Applying integration by parts four times and taking into account the properties of the functions $\omega(x)$ and $X_n(x)$, we get

$$\lambda_n^{3/2}\omega_n = \lambda_n^{1/2}\int_0^1\left[x^{\beta}\omega_n'(x)\right]' X_n(x)dx = \lambda_n^{1/2}\int_0^1\left[L\omega(x)\right]X_n(x)dx.$$

Replacing the function X_n in the last integral with $c\left[x^{\beta}X_n'(x)\right]'/\lambda_n$ and then applying integration by parts twice to the resulting integral, we have:

$$\lambda_n^{3/2}\omega_n = \int_0^1 x^{\beta/2}\left[L\omega(x)\right]'\left\{\lambda_n^{-1/2}x^{\beta/2}X_n'(x)\right\}dx.$$

Thus, the number $\lambda_n^{3/2}\omega_n$ is the Fourier coefficient of the function $x^{\beta/2}\left[L\omega(x)\right]'$ concerning the orthonormal system of functions $\left\{\lambda_n^{-1/2}x^{\beta/2}X_n'(x)\right\}\Big|_{n=1}^{+\infty}$. Then according to Bessel's inequality, inequality (13) holds.

Lemma 4 has been proved.

3.4. Existence of the solution

In this subsection, we will prove the existence of the solution to the problem. For this aim, first, we will expand the functions $f(x)$, $\varphi(x)$ and $\psi(x)$ by the series concerning the system of functions $\{X_n\}_{n=1}^{\infty}$:

$$\varphi(x) = \sum_{n=1}^{\infty}\varphi_n X_n(x),\ f(x) = \sum_{n=1}^{\infty} f_n X_n(x),\ \psi(x) = \sum_{n=1}^{+\infty}\psi_n X_n(x), \qquad (14)$$

where φ_n, f_n and ψ_n are the Fourier coefficients of the functions $f(x)$, $\varphi(x)$ and $\psi(x)$, respectively, defined by

$$\varphi_n = \int_0^1 \varphi(x)X_n(x)dx,\ f_n = \int_0^1 f(x)X_n(x)dx,\ \psi(x) = \int_0^1 \psi_n X_n(x)dx.$$

We will seek a solution to the Problem, in the following form

$$u(x,t)=\sum_{n=1}^{\infty}u_n(t)X_n(x), \tag{15}$$

where $u_n(t)$ is an unknown function.

Substituting (14) and (15) into equation (1), the initial condition (2) and the over-determination condition (3), we obtain the following problem for determining the unknown functions $u_n(t)$ and f_n:

$$\begin{cases} {}_C D_{0t}^{\alpha} u_n(t)+\lambda_n u_n(t)=f_n, \\ u_n(0)=\varphi_n, \\ u_n(T)=\psi_n. \end{cases}$$

It is easy to verify that from the last problem, we can write $u_n(t)$ and f_n in the following forms:

$$u_n(t)=\frac{1-E_{\alpha,1}(-\lambda_n t^{\alpha})}{1-E_{\alpha,1}(-\lambda_n T^{\alpha})}(\varphi_n-\psi_n)+\varphi_n, \tag{16}$$

$$f_n=\varphi_n\lambda_n-\frac{\lambda_n(\varphi_n-\psi_n)}{1-E_{\alpha,1}(-\lambda_n T^{\alpha})}. \tag{17}$$

Then, the formal solution to the inverse source problem is given by

$$u(x,t)=\sum_{n=1}^{+\infty}\left(\frac{1-E_{\alpha,1}(-\lambda_n t^{\alpha})}{1-E_{\alpha,1}(-\lambda_n T^{\alpha})}(\varphi_n-\psi_n)+\varphi_n\right)X_n(x), \tag{18}$$

$$f(x)=\sum_{n=1}^{+\infty}\left(\varphi_n\lambda_n-\frac{\lambda_n(\varphi_n-\psi_n)}{1-E_{\alpha,1}(-\lambda_n T^{\alpha})}\right)X_n(x). \tag{19}$$

Theorem 1. *Let the functions φ and ψ satisfy the conditions Lemma 4. Then, the series defined by (18) and (19) will be the unique solution to the inverse source problem.*

Proof. First, we consider the series (18). It is known that for $0<\alpha<1$ the following estimate is valid for the Mittag-Leffler function[4]:

$$\frac{1}{1+\Gamma(1-\alpha)z}\le E_{\alpha,1}(-z)\le\frac{1}{1+\Gamma(1+\alpha)^{-1}z}, \quad z>0.$$

Considering the above estimate, from (18), one can easily get

$$|u(x,t)|\le\sum_{n=1}^{+\infty}\left(\left|\frac{1-E_{\alpha,1}(-\lambda_n t^{\alpha})}{1-E_{\alpha,1}(-\lambda_n T^{\alpha})}\right||\varphi_n-\psi_n|+|\varphi_n|\right)|X_n(x)|\le$$

$$\le 2\sum_{n=1}^{+\infty}|\varphi_n||X_n(x)|+\sum_{n=1}^{+\infty}|\psi_n||X_n(x)|=2\sum_{n=1}^{+\infty}\sqrt{\lambda_n}|\varphi_n|\frac{|X_n(x)|}{\sqrt{\lambda_n}}+\sum_{n=1}^{+\infty}\sqrt{\lambda_n}|\psi_n|\frac{|X_n(x)|}{\sqrt{\lambda_n}}.$$

Hence, applying the Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{+\infty}\sqrt{\lambda_n}|\varphi_n|\frac{|X_n(x)|}{\sqrt{\lambda_n}}\le\left(\sum_{n=1}^{+\infty}\lambda_n\varphi_n^2\sum_{n=1}^{+\infty}\frac{X_n^2(x)}{\lambda_n}\right)^{\frac{1}{2}},$$

$$\sum_{n=1}^{+\infty}\sqrt{\lambda_n}\left|\psi_n\right|\frac{\left|X_n(x)\right|}{\sqrt{\lambda_n}}\leq\left(\sum_{n=1}^{+\infty}\lambda_n\psi_n^2\sum_{n=1}^{+\infty}\frac{X_n^2(x)}{\lambda_n}\right)^{\frac{1}{2}}.$$

The series on the right-hand side of these inequalities converges uniformly in $x\in[0,1]$ due to Lemma 2 and Lemma 3. Hence, the series on the left-hand side also converges uniformly in x on $[0,1]$. Therefore, the series (18) will be absolutely and uniformly convergent in $\overline{\Omega}$.

Now, we consider the series corresponding to the function $\left[x^\beta u_x\right]_x$. From (18), considering (5), we get

$$\left|\left[x^\beta u_x\right]_x\right|\leq\sum_{n=1}^{+\infty}\left|\frac{1-E_{\alpha,1}(-\lambda_n t^\alpha)}{1-E_{\alpha,1}(-\lambda_n T^\alpha)}(\varphi_n-\psi_n)+\varphi_n\right|\left|\left[x^\beta X_n'(x)\right]'\right|=$$

$$=\sum_{n=1}^{+\infty}\lambda_n\left(\left|\frac{1-E_{\alpha,1}(-\lambda_n t^\alpha)}{1-E_{\alpha,1}(-\lambda_n T^\alpha)}\right|\left|(\varphi_n-\psi_n)\right|+\left|\varphi_n\right|\right)\left|X_n(x)\right|\leq2\sum_{n=1}^{+\infty}\lambda_n\left|\varphi_n\right|\left|X_n(x)\right|+\sum_{n=1}^{+\infty}\lambda_n\left|\psi_n\right|\left|X_n(x)\right|$$

$$=2\sum_{n=1}^{+\infty}\lambda_n^{3/2}\left|\varphi_n\right|\frac{\left|X_n(x)\right|}{\sqrt{\lambda_n}}+\sum_{n=1}^{+\infty}\lambda_n^{3/2}\left|\psi_n\right|\frac{\left|X_n(x)\right|}{\sqrt{\lambda_n}}.$$

From here, using the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=1}^{+\infty}\lambda_n^{3/2}\left|\varphi_n\right|\frac{\left|X_n(x)\right|}{\sqrt{\lambda_n}}\leq\left(\sum_{n=1}^{+\infty}\lambda_n^3\varphi_n^2\sum_{n=1}^{+\infty}\frac{X_n^2(x)}{\lambda_n}\right)^{\frac{1}{2}},$$

$$\sum_{n=1}^{+\infty}\lambda_n^{3/2}\left|\psi_n\right|\frac{\left|X_n(x)\right|}{\sqrt{\lambda_n}}\leq\left(\sum_{n=1}^{+\infty}\lambda_n^3\psi_n^2\sum_{n=1}^{+\infty}\frac{X_n^2(x)}{\lambda_n}\right)^{\frac{1}{2}}.$$

The series on the right-hand side of these inequalities converges uniformly in $x\in[0,1]$ due to Lemma 2 and Lemma 4. Hence, the series on the left-hand side also converges uniformly in x on $[0,1]$. Therefore, the series corresponding to the function $\left[x^\beta u_x\right]_x$ will be absolutely and uniformly convergent in any compact $D\subset\Omega$.

Now, considering the series corresponding to the function f from (19), we get

$$\left|f(x)\right|\leq\sum_{n=1}^{+\infty}\left|\lambda_n\varphi_n X_n(x)\right|+\sum_{n=1}^{+\infty}\frac{\lambda_n\left|\varphi_n-\psi_n\right|}{1-E_{\alpha,1}(-\lambda_n T^\alpha)}\left|X_n(x)\right|\leq$$

$$\leq2\sum_{n=1}^{+\infty}\left|\varphi_n\right|\left|X_n(x)\right|+\sum_{n=1}^{+\infty}\left|\psi_n\right|\left|X_n(x)\right|.$$

Similarly, one can show convergence of these series under the assumptions of Theorem 1.

3.5. Uniqueness of the solution of the problem

Now, we prove the uniqueness of the solution to the problem. For this aim, we suppose that $\{u_1,f_1\}$ and $\{u_2,f_2\}$ are solutions to the problem. Define their difference as

$$u(x,t)=u_1(x,t)-u_2(x,t),\ f(x)=f_1(x)-f_2(x).$$

Since both functions satisfy equation (1) in the domain Ω, their difference $\{u(x,t),f(x)\}$ also satisfies the same equation, and on the boundary, $u(x,t)$ satisfies the following conditions:

$$u(x,0)=0,\ u(x,T)=0,$$

$$pu(0,t)=qu(1,t),\ q\left[x^{\beta}u_x\right]\Big|_{x=0}=p\left[x^{\beta}u_x\right]\Big|_{x=1}.$$

Now, consider the function:

$$u_n(t)=\int_0^1 u(x,t)X_n(x)dx. \tag{20}$$

This equation can be rewritten as:

$${}_C D_{0t}^{\alpha}u_n(t)=\int_0^1 {}_C D_{0t}^{\alpha}u(x,t)X_n(x)dx. \tag{21}$$

Using the equation (1), we further transform equation (21) into:

$${}_C D_{0t}^{\alpha}u_n(t)=\int_0^1\left\{\left[x^{\beta}u_x(x,t)\right]_x+f(x)\right\}X_n(x)dx \tag{23}$$

after performing some simplifications, we get

$${}_C D_{0t}^{\alpha}u_n(t)=\int_0^1\left[x^{\beta}u_x(x,t)\right]_x X_n(x)dx+f_n. \tag{24}$$

Using (20) and (24), along with the conditions $u(x,0)=0$ and $u(x,T)=0$, we obtain

$$\begin{cases} {}_C D_{0t}^{\alpha}u_n(t)+\lambda_n u_n(t)=f_n,\\ u_n(0)=0,\\ u_n(T)=0.\end{cases}$$

From the solution, it is clear $f_n=0$, $u_n(t)\equiv 0$. Then from the last equalities taken into account (15), it follows that

$$\int_0^1 u(x,t)X_n(x)dx=0. \tag{25}$$

As since problem {(5),(6)} is self-adjoint, its eigenfunctions will be a complete system in $L_2[0,1]$. Considering this from (25), we conclude that $u(x,t)\equiv 0$ a.e. on $[0,1]$ for all $t\in[0,T]$. Under $u(x,t)\in C(\overline{\Omega})$, we obtain $u(x,t)\equiv 0$ in $\overline{\Omega}$. Thus, a homogeneous problem has only a trivial solution and that problem has a unique solution.

CONCLUSION

In the present paper, we consider an inverse-source problem for a second-order, time-fractional, space-degenerate partial differential equation in a rectangular domain. By applying the method of separation of variables, we obtain a spectral problem for an ordinary differential equation. Since this equation has a degenerate coefficient, we cannot find the eigenvalues and eigenfunctions of this spectral problem in explicit form. However, by applying the theory of integral equations with symmetric kernels, we establish the existence and some properties of the eigenvalues and eigenfunctions of this spectral problem. Using these properties, we prove the existence and uniqueness of the solution of the main problem.

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