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SOHA CHEGARASIDA BUZILADIGAN TO‘RTINCHI TENGLAMA UCHUN ARALASH MASALA

СМЕШАННАЯ ЗАДАЧА ДЛЯ УРАВНЕНИЯ ЧЕТВЕРТОГО ПОРЯДКА, ВЫРОЖДАЮЩЕГОСЯ НА ГРАНИЦЫ ОБЛАСТИ

A MIXED PROBLEM FOR A FOURTH ORDER EQUATION DEGENERATING ON THE PART OF THE BOUND OF THE DOMAIN

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Annotatsiya

Ushbu maqolada to‘rtburchak sohada Kaputo ma’nosidagi kasr hosilasini o‘z ichiga olgan to‘rtinchi tartibli bitta buziladigan tenglama uchun boshlang'ich-chegaraviy masala bayon qilingan va o‘rganilgan. Masala yechimining mavjudligi va yagonaligi isbotlangan. O‘zgaruvchilarni ajratish usuli orqali, oddiy differensial tenglama uchun spektral masala hosil qilingan. Bu spektral masala Grin funksiyasi yordamida simmetrik yadroli ikkinchi tur Fredgolm integral tenglamasiga ekvivalent keltirilgan. O‘rganilayotgan masalaning yechimi spektral masalaning xos funksiyalar sistemasiga nisbatan Furye qatorining yig‘indisi sifatida topilgan.

Аннотация

В данной статье для одного вырождающегося уравнения четвертого порядка, содержащего дробную производную Капуто, в прямоугольной области формулируется и исследуется начально-краевая задача. Доказаны существование и единственность решения задачи. В то же время путем применения метода разделения переменных к рассматриваемой задаче получена спектральная задача для обыкновенного дифференциального уравнения. Далее строится функция Грина спектральной задачи, с помощью которой она эквивалентно сводится к интегральному уравнению Фредгольма второго рода с симметричным ядром. Решение рассматриваемой задачи записано в виде суммы ряда Фурье по системе собственных функций спектральной задачи.

Abstract

In this article, for one degenerate equation of the fourth order, containing the fractional derivative of Caputo, in rectangular domain, a initial-boundary problem is formulated and investigated. The existenceand uniqueness of the solution of the problem have been proved. At the same time, by applying the method of separation of variables to the considered problem, a spectral problem for an ordinary differential equation has been obtained. Next, the Green's function of the spectral problem was constructed, with the help of which it is equivalently reduced to an the second kind Fredholm integral equation with a symmetric kernel. The solution of the considered problem has been written as the sum of a Fourier series with respect to the system of eigenfunctions of the spectral problem.

Kalit so‘zlar: *buziladigan to‘rtinchi tartibli tenglama; chegaraviy masala; spektral masala; Grin funksiyasi; integral tenglama; yechimning mavjudligi va yagonaligi.*

Ключевые слова: *вырождающееся уравнение четвертого порядка, краевая задача, спектральная задача, функция Грина, интегральное уравнение, существование и единственность решения.*

Key words: *degenerate fourth order equation; initial-boundary value problem; spectral problem; Green's function; integral equation; existence and uniqueness of the solution.*

I. I NTRODUCTION

It is known that the theory of differential equations has a long and rich history. Until the last quarter of the twentieth century in this theory integer-order differential equations were considered. In connection with the development of fractional (differential and integral) analysis beginning in the late twentieth century, researchers began to deal with differential equations containing fractional derivatives. At present, numerous scientific articles have been published that consider initial, boundary and spectral problems for differential equations (both ordinary and partial derivatives), containing fractional derivatives with various modifications (see e.g. [1] - [4] and [5], as well as the cited there literature). The books [6] and [7] played a significant role in the development of this trend. Below we provide a brief overview research (close topics of this article) by a fourth-order partial differential equation containing a fractional derivative of an unknown function by temporary variable.

In the papers [8] – [10], initial boundary value problems for a one-dimensional and two-dimensional fourth-order equation containing the operator fractional differentiation of the Caputo with respect to the time variable, and in [10] the inverse problem is also considered. Initial-boundary problems also studied for fourth-order equations with fractional differentiation operator Hilfer, Dzhribashyan-Nersesyan and Riemann-Liouville, respectively, in the papers [11], [12] and [13]. The direct and inverse problems for a mixed-type fourth-order equation with the Hilfer operator are studied in [14] and [15], respectively. In this direction, we also note the works [17] and [18], where inverse problems of determining the order of a fractional derivative in the sense of respectively Riemann-Liouville and Caputo into the subdiffusion equation and the wave equation with an arbitrary positive operator, having a discrete spectrum.

In the papers listed above, only non-degenerate equations were considered. But both local and nonlocal boundary value problems for degenerate partial differential equations containing fractional derivatives of an unknown function remain unexplored. The study of boundary value problems for such equations is of great importance not only from a theoretical point of view, but also from a practical one, because such equations and problems for them arise in mathematical modeling of many problems in the theory of gas and hydrodynamics, the theory of small bendings of surfaces, mathematical biology and other branches of science.

Initial-boundary value problems for degenerate equations with fourth-order frequent derivatives containing the first and second time derivatives were previously studied in [18] - [20].

II. STATEMENT OF THE PROBLEM

In the domain $\Omega=\{(x,t):0<x<l,0<t<T\}$ we consider the following equation

$$ {}_C D_{0t}^{\alpha} u(x,t)+\left[x^{\beta} u_{xx}(x,t)\right]_{xx}=f(x,t), \quad (1)$$

where ${}_C D_{0t}^{\alpha}$ is Caputo fractional differential operator α order [1]

$$ {}_C D_{0t}^{\gamma} g(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{g^{(n)}(z)\,dz}{(t-z)^{\gamma-n+1}}, \quad \left(n=\mathrm{Re}(\gamma)+1, t>0\right)$$

α, β, l, T are given real numbers, such that $0<\alpha\le 1$, $0\le\beta<1$, $l,T>0$; $f(x,t)$ is a given function on Ω.

A function $u(x,t)$ satisfying in the domain Ω equation (1) and the following conditions $u(x,t)\in C(\overline{\Omega})$, ${}_C D_{0t}^{\alpha} u(x,t)$, $\left(x^{\beta} u_{xx}\right)_{xx}\in C(\Omega)$ is called regular in the domain Ω solution of the equation (1).

For the equation (1) in the domain Ω we investigate the following mixed problem:

Problem. *Find a regular in the domain Ω solution of the equation (1) satisfying the following initial condition*

$$u(x,0)=\varphi(x), \quad 0\le x\le l \quad (2)$$

and the following boundary conditions:

$$u(0,t)=0,\ 0\le t\le T,\ u_x(0,t)=u_{xx}(l,t)=u_{xxx}(l,t)=0,\ 0<t\le T, \quad (3)$$

where $\varphi(x)$ is a given function continuous on $[0,l]$, such that $\varphi(0)=0$.

Firstly, we consider homogeneous equation

$$ {}_C D_{0t}^{\alpha} u(x,t)=\left[x^{\beta} u_{xx}(x,t)\right]_{xx}, \quad (x,t)\in\Omega.$$

We seek solution of the equation (1) satisfying condition in the form $u(x,t)=v(x)T(t)$. Then, with respect to the function $T(t)$, we get the following equation

$$ {}_C D_{0t}^{\alpha} T(t) - \lambda T(t) = 0, $$

and with respect to $v(x)$, we get the following spectral problem:

$$ Lv = \left[x^{\beta} v''(x) \right]'' = \lambda v(x), \qquad x \in (0,l); \tag{4} $$

$$ v(0) = v'(0) = v''(l) = v'''(l) = 0, \tag{5} $$

i.e. a problem finding those values of the parameter λ for which there exist nontrivial solution of equation (4) satisfying conditions (5).

Let's investigate problem {(4),(5)}. At first, we define the sign of the λ. To do this,multiplying the both sides of (4) to the function $v(x)$ and integrating on $[0,l]$, we obtain

$$ \lambda \int_0^l v^2(x)dx = \int_0^l \left[x^{\beta} v''(x) \right]'' v(x)dx. $$

Using the rule of integration by parts two times on the right part of the last equality, we get

$$ \lambda \int_0^l v^2(x)dx = \left[\left[x^{\beta} v''(x) \right]' v(x) - x^{\beta} v''(x) v'(x) \right]\Big|_0^l + \int_0^l x^{\beta} \left[v''(x) \right]^2 dx. $$

From here, taking into account condition (5), we have

$$ \lambda \int_0^l v^2(x)dx = \int_0^l x^{\beta} \left[v''(x) \right]^2 dx, $$

from which it follows that $\lambda \ge 0$.

Let be $\lambda = 0$, i.e. consider the equation $\left[x^{\beta} v''(x) \right]'' = 0$. The general solution of this equation has the form

$$ v(x) = C_1 \frac{x^{3-\beta}}{(2-\beta)(3-\beta)} + C_2 \frac{x^{2-\beta}}{(1-\beta)(2-\beta)} + C_3 x + C_4, $$

where C_j, $j = \overline{1,4}$ are arbitrary constants.

Satisfying this function to the conditions (5), we get

$$ C_4 = 0,\ C_3 = 0,\ C_1 l + C_2 = 0,\ C_1(1-\beta)l - C_2 \beta = 0, $$

from which it follows that $C_1 = C_2 = C_3 = C_4 = 0$. Hence, $v(x) \equiv 0$. From the above proved it follows that problem {(4),(5)} can have nontrivial solution only for $\lambda > 0$.

Now we are engaged in the proof of the existence of eigenvalues and eigenfunctions of the problem the problem {(4),(5)}. For this aim we construct Green function of the problem {(4),(5)}. It is unique and represents as follows [Байкузиев. С. 181]:

$$ G(x,s) = \begin{cases} \dfrac{x^{3-\beta}}{(2-\beta)(3-\beta)} - \dfrac{s x^{2-\beta}}{(1-\beta)(2-\beta)}, & x < s, \\ \dfrac{s^{3-\beta}}{(2-\beta)(3-\beta)} - \dfrac{x s^{2-\beta}}{(1-\beta)(2-\beta)}, & x > s. \end{cases} $$

Then the problem {(4),(5)} will be equivalent to the integral equation [1]

$$ v(x) = \lambda \int_0^l G(x,s) v(s) ds. \tag{6} $$

Since $G(x,s)$ is symmetric and continuous kernel, then from the theory of integral equations with symmetric kernels [2] it follows that equation (6) [hence problem {(4),(5)}] has a countable set of eigenvalues

$$0<\lambda_1<\lambda_2<\lambda_3<\ldots$$

and corresponding orthonormal eigenfunctions

$$v_1(x), v_2(x), v_3(x), \ldots,$$

and also an arbitrary function $g(x)\in L_2[0,l]$ expands into a series in these eigenfunctions, which converges on average.

III. CONVERGENCE OF BASIC BILINEAR

Lemma 1. On the interval $[0,l]$ *the following series converge uniformly:*

$$\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k},\quad \sum_{k=1}^{+\infty}\frac{\left[v_k'(x)\right]^2}{\lambda_k},\quad \sum_{k=1}^{+\infty}\frac{\left[x^\beta v_k''(x)\right]^2}{\lambda_k^2},\quad \sum_{k=1}^{+\infty}\frac{\left\{[x^\beta v_k''(x)]'\right\}^2}{\lambda_k^2}.$$

Proof. Since for $0\le\beta<1$ the kernel $G(x,s)$ of integral equation (6) is symmetric, continuous and positive-definite, on the basis of the Mercer's theorem (см. И.Г. Петровский [3]), we get

$$G(x,s)=\sum_{k=1}^{+\infty}\frac{v_k(x)v_k(s)}{\lambda_k},$$

particularly

$$\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}=G(x,x)\le const. \tag{7}$$

Hence, it follows that the series $\sum_{k=1}^{+\infty}\left[v_k^2(x)/\lambda_k\right]$ converges uniformly.

From the integral equation (6), taking continuity of the function $G(x,s)$ into account, we get

$$v_k'(x)=\lambda_k\int_0^l\frac{\partial G(x,s)}{\partial x}v_k(s)ds$$

or

$$v_k'(x)=\lambda_k\int_0^x\frac{\partial G(x,s)}{\partial x}v_k(s)ds+\lambda_k\int_x^l\frac{\partial G(x,s)}{\partial x}v_k(s)ds.$$

Using equation (4), from the last, we obtain

$$v_k'(x)=\int_0^x\frac{\partial G(x,s)}{\partial x}\left[s^\beta v_k''(s)\right]''ds+\int_x^l\frac{\partial G(x,s)}{\partial x}\left[s^\beta v_k''(s)\right]''ds.$$

From here, applying the rule of integrating by parts two times, we get

$$v_k'(x)=\left[\frac{\partial G(x,s)}{\partial x}\left[s^\beta v_k''(s)\right]'-\frac{\partial^2 G(x,s)}{\partial x\partial s}s^\beta v_k''(s)\right]_{s=0}^{s=x}+$$

$$+\left[\frac{\partial G(x,s)}{\partial x}\left[s^{\beta}v_k''(s)\right]' - \frac{\partial^2 G(x,s)}{\partial x\partial s}s^{\beta}v_k''(s)\right]_{s=x}^{s=l} + \int_0^l \frac{\partial^3 G(x,s)}{\partial x\partial s^2}s^{\beta}v_k''(s)ds.$$

Taking into account $\frac{\partial G(x,0)}{\partial x}=0$, $\frac{\partial^2 G(x,0)}{\partial x\partial s}=0$, $v_k''(l)=0$, $v_k'''(l)=0$ and the continuity of the functions $\frac{\partial G(x,s)}{\partial x}$, $\frac{\partial^2 G(x,s)}{\partial x\partial s}$ for $x=s$, $v_k''(s)$, $\left[s^{\beta}v_k''(s)\right]'$, we write the last equality in the form

$$v_k'(x)=\int_0^l \frac{\partial^3 G(x,s)}{\partial x\partial s^2}s^{\beta}v_k''(s)ds=\sqrt{\lambda_k}\int_0^l\left[s^{\beta/2}\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\right]\left[\frac{s^{\beta/2}v_k''(s)}{\sqrt{\lambda_k}}\right]ds$$

or

$$\frac{v_k'(x)}{\sqrt{\lambda_k}}=\int_0^l\left[s^{\beta/2}\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\right]\left[\frac{s^{\beta/2}v_k''(s)}{\sqrt{\lambda_k}}\right]ds. \qquad (8)$$

It is easy to show that $\left\{s^{\beta/2}v_k''(s)\big/\sqrt{\lambda_k}\right\}\Big|_{k=1}^{+\infty}$ is orthonormal system. Indeed,

$$\left(\lambda_k\lambda_m\right)^{-1/2}\int_0^l s^{\beta/2}v_k''(s)\cdot s^{\beta/2}v_m''(s)ds=\left(\lambda_k\lambda_m\right)^{-1/2}\int_0^l s^{\beta}v_k''(s)v_m''(s)ds=$$

$$=\left(\lambda_k\lambda_m\right)^{-1/2}\left\{s^{\beta}v_k''(s)v_m''(s)-\left[s^{\beta}v_k''(s)\right]'v_m(s)\right\}\Big|_0^l+\left(\lambda_k\lambda_m\right)^{-1/2}\int_0^l\left[s^{\beta}v_k''(s)\right]''v_m(s)ds=$$

$$=\sqrt{\lambda_k/\lambda_m}\int_0^l v_k(s)v_m(s)ds=\begin{cases}0, & npu\ k\neq m,\\ 1, & npu\ k=m.\end{cases}.$$

Then from (8) follows that function $v_k'(x)\big/\sqrt{\lambda_k}$ can be considered as the Fourier coefficient of the function $s^{\beta/2}\frac{\partial^3 G(x,s)}{\partial x\partial s^2}$ with respect to variable s.

If for all x $s^{\beta/2}\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\in L_2[0,l]$, then based on Bessel's inequality, we obtain

$$\sum_{k=1}^{+\infty}\frac{\left[v_k'(x)\right]^2}{\lambda_k}\le\int_0^l s^{\beta}\left[\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\right]^2 ds. \qquad (9)$$

We shall prove that the right hand part of the (9) is limited:

$$\int_0^l s^{\beta}\left[\frac{\partial^3 G(x,s)}{\partial x\partial s^2}\right]^2 ds=\int_0^x s^{\beta}\left\{\frac{\partial^3}{\partial x\partial s^2}\left[\frac{x^{3-\beta}}{(2-\beta)(3-\beta)}-\frac{sx^{2-\beta}}{(1-\beta)(2-\beta)}\right]\right\}^2 ds+$$

$$+\int_x^l s^{\beta}\left\{\frac{\partial^3}{\partial x\partial s^2}\left[\frac{s^{3-\beta}}{(2-\beta)(3-\beta)}-\frac{xs^{2-\beta}}{(1-\beta)(2-\beta)}\right]\right\}^2 ds=\int_x^l s^{\beta}\left(-s^{-\beta}\right)^2 ds=$$

$$=\int_x^l s^{-\beta}ds=\frac{l^{1-\beta}-x^{1-\alpha}}{1-\alpha}\le\frac{l^{1-\beta}}{1-\alpha}<+\infty .$$

From the last we conclude that series $\sum_{k=1}^{+\infty}\left\{\left[v'(x)\right]^2/\lambda_k\right\}$ converges uniformly.

Further, from the equation (6), taking into account the function $\frac{\partial G(x,s)}{\partial x}$ is continuous, we get

$$x^\beta v_k''(x)=\lambda_k\int_0^l x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}v_k(s)ds \qquad (10)$$

or

$$\frac{x^\beta v_k''(x)}{\lambda_k}=\int_0^l x^\beta\frac{\partial^2 G(x,s)}{\partial s^2}v_k(s)ds .$$

From here considering $\left[x^\beta v_k''(x)/\lambda_k\right]$ as the Fourier coefficient of the $x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}$, based on Bessel's equality, we obtain

$$\sum_{k=1}^{+\infty}\frac{\left[x^\beta v_k''(x)\right]^2}{\lambda_k^2}\le\int_0^l\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]^2ds . \qquad (11)$$

We show that the right-hand part of the last equality is limited:

$$\int_0^l\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]^2ds=\int_0^x\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]^2ds+\int_x^l\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]^2ds=$$

$$=\int_0^x x^{2\beta}\left(x^{1-\beta}-sx^{-\beta}\right)^2ds=\int_0^x x^{2\beta}\left(x^{2-2\beta}-2sx^{1-2\beta}+s^2x^{-2\beta}\right)ds=$$

$$=\int_0^x\left(x^2-2sx+s^2\right)ds=\frac{1}{3}x^3<\frac{l^3}{3}.$$

Hence, series $\sum_{k=1}^{+\infty}\left\{\left[x^\beta v_k''(x)\right]^2/\lambda_k^2\right\}$ converges uniformly.

Similarly, from equation (5), taking into account the function $x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}$ is continuous, we get

$$\frac{\left[x^\beta v_k''(x)\right]'}{\lambda_k}=\int_0^l\frac{\partial}{\partial x}\left[x^\beta\frac{\partial^2 G(x,s)}{\partial x^2}\right]v_k(s)ds .$$

Hence, on the basis of Bessel's equality, we have

$$\sum_{k=1}^{+\infty} \frac{\left[\left(x^{\beta} v_k''(x)\right)'\right]^2}{\lambda_k^2} \le \int_0^l \left\{ \frac{\partial}{\partial x}\left[x^{\beta} \frac{\partial^2 G(x,s)}{\partial x^2} \right] \right\}^2 ds. \tag{12}$$

We show that the right hand part of (12) is limited:

$$\int_0^l \left\{ \frac{\partial}{\partial x}\left[x^{\beta} \frac{\partial^2 G(x,s)}{\partial x^2} \right] \right\}^2 ds = \int_0^x \left\{ \frac{\partial}{\partial x}\left[x^{\beta}\left(x^{1-\beta} - s x^{-\beta} \right) \right] \right\}^2 ds = \int_0^x \left[\frac{\partial}{\partial x}(x-s) \right]^2 ds = x < l.$$

Hence it follows that the series $\sum_{k=1}^{+\infty} \left\{ \left[x^{\beta} v_k''(x) \right]^2 / \lambda_k^2 \right\}$ converges uniformly.

IV. THE ORDER OF THE FOURIER COEFFICIENTS

Lemma 1. *Suppose the following conditions hold:*

$f(x),\ f'(x) \in C[0,l],\ x^{\beta/2} f''(x) \in L_2[0,l],\ f(0) = f'(0) = 0.$

Then the following Bessel type equality is valid:

$$\sum_{k=1}^{\infty} \lambda_k f_k^2 \le \int_0^l x^{\beta} \left[f''(x) \right]^2 dx. \tag{13}$$

Particularly, it is possible to assert convergence of the series in (13).

Here in after f_k denotes Fourier coefficient of the function $f(x)$ by the system of eigenfunctions $v_k(x)$.

Proof. Let's consider the functional

$$J = \int_0^l x^{\beta} \left\{ \left[f(x) - \sum_{k=1}^{n-1} f_k v_k(x) \right]'' \right\}^2 dx = \int_0^l x^{\beta} \left[f''(x) \right]^2 dx + \int_0^l x^{\beta} \left[\sum_{k=1}^{n-1} f_k v_k''(x) \right]^2 dx -$$

$$-2\int_0^l x^{\beta} f''(x) \left[\sum_{k=1}^{n-1} f_k v_k''(x) \right] dx = \int_0^l x^{\beta} \left[f''(x) \right]^2 dx + \sum_{k=1}^{n-1} f_k^2 \int_0^l x^{\beta} \left[v_k''(x) \right]^2 dx +$$

$$+2 \sum_{\substack{k,l=1 \\ k \ne l}}^{n-1} f_k f_l \int_0^l x^{\beta} v_k''(x) v_l''(x) dx - 2 \sum_{k=1}^{n-1} f_k \int_0^l x^{\beta} f''(x) v_k''(x) dx. \tag{14}$$

Integrating by parts twice, we get

$$\int_0^l x^{\beta} v_k''(x) v_l''(x) dx = \left[x^{\beta} v_k''(x) v_l'(x) - \left[x^{\beta} v_k''(x) \right]' v_l(x) \right]_{x=0}^{x=l} +$$

$$+\int_0^l \left[x^{\beta} v_k''(x) \right]'' v_l(x) dx = \lambda_k \int_0^l v_k(x) v_l(x) dx = \begin{cases} 0 & npu \quad k \ne l, \\ \lambda_k & npu \quad k = l. \end{cases} \tag{15}$$

Similarly integrating by parts, we obtain

$$\int_0^l x^{\beta} f''(x) v_k''(x) dx = \left[x^{\beta} v_k''(x) f'(x) - \left[x^{\beta} v_k''(x) \right]' f(x) \right]_{x=0}^{x=l} +$$

$$+\int_0^l \left[x^\beta v_k''(x)\right]'' f(x)dx = \lambda_k \int_0^l f(x)v_k(x)dx = \lambda_k f_k. \quad (16)$$

By virtue of (15) and (16), from (14) it follows that

$$J = \int_0^l x^\beta \left[f''(x)\right]^2 dx - \sum_{k=1}^{n-1} \lambda_k f_k^2 \geq 0 \text{ for } \forall n \in N.$$

From the last inequality, it follows that the fairness of the inequality (14).

Lemma 2. Suppose that the following conditions hold:

$$f(x),\ f'(x),\ x^\beta f''(x),\ \left[x^\beta f''(x)\right]' \in C[0,l],\ \left[x^\beta f''(x)\right]'' \in L_2[0,l],$$

$$f(0) = f'(0) = f''(l) = f'''(l) = 0.$$

Then the following Bessel type equality is valid:

$$\sum_{k=1}^{\infty} \lambda_k^2 f_k^2 \leq \int_0^l \left[\left(x^\beta f''(x)\right)''\right]^2 dx. \quad (17)$$

Particularly, it is possible to assert convergence of the series in (17).

Proof. By virtue of the conditions of lemma 2 equality (16) is fulfilled. In addition, the following equality is valid as

$$\int_0^l x^\beta f''(x) v_k''(x)dx = \int_0^l \left[x^\beta f''(x)\right]'' v_k(x)dx. \quad (18)$$

Indeed, integrating by parts the left hand part of the last equality gives

$$\int_0^l x^\beta v_k''(x) f''(x)dx = \left\{x^\beta f''(x)v_k'(x) - \left[x^\beta f''(x)\right]' v_k(x)\right\}\Big|_{x=0}^{x=l} +$$

$$+\int_0^l \left[x^\beta f''(x)\right]'' v_k(x)dx.$$

Taking into account $x^\beta f''(x),\ \left[x^\beta f''(x)\right]'$ are continuous on $[0,l]$ and boundary conditions $v_k(0) = v_k'(0) = 0$, and also conditions $f''(l) = f'''(l) = 0$, we get

$$\left[x^\beta f''(x)v_k''(x) - \left[x^\beta f_k''(x)\right]' v_k(x)\right]_{x=0}^{x=l} = 0.$$

From (16) and (18) it follows that

$$\int_0^1 \left[x^\alpha f''(x)\right]'' v_k(x)dx = \lambda_k f_k.$$

Hence, $\lambda_k f_k$ is the Fourier coefficient of the function $Lf \equiv \left[x^\beta f''(x)\right]''$, or

$$(Lf)_k = \left[\left(x^\beta f''(x)\right)''\right]_k = \lambda_k f_k. \quad (19)$$

Writing Bessel's inequality for function $\left[x^{\beta} f''(x)\right]''$ and taking equality (19) into account, we get sought inequality.

V. Justification of the method of Fourier

Theorem 1. Suppose that the following conditions hold:

1. $\varphi(x)$ satisfies conditions of the lemma 1.

2. $f(x,t)$ satisfies conditions of the lemma 1 uniformly with respect to t

3. $f_t'(x,t)$ satisfies conditions of the lemmas 1 and 2 uniformly with respect to variable t.

Then the following

$$u(x,t)=\sum_{k=1}^{+\infty}\left[\varphi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)+\int_0^t (t-z)^{\alpha-1} E_{\alpha,\alpha}\left[-\lambda_k (t-z)^{\alpha}\right] f_k(z)\,dz\right] v_k(x) \quad (20)$$

series will be regular solution of the problem $\{(1),(2),(3)\}$, where $E_{\alpha,\beta}(z)$ is two-parameter Mittage-Leffler's function[]:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}.$$

Proof. We shall seek solution of the considered in the form

$$u(x,t)=\sum_{k=1}^{\infty} T_k(t) v_k(x) \quad (21)$$

where $v_k(x)$, $k\in N$ are functions which are defined by (7), and $T_k(t)$, $k\in N$ are unknown functions.

Expanding functions $f(x,t)$ and $\varphi(x)$ to series by functions $v_k(x)$, we get

$$f(x,t)=\sum_{k=1}^{\infty} f_k(t) v_k(x),\ \varphi(x)=\sum_{k=1}^{\infty}\varphi_k v_k(x) \quad (22)$$

where

$$f_k(t)=\int_0^l f(x,t) v_k(x)\,dx,\ \ \varphi_k=\int_0^l \varphi(x) v_k(x)\,dx. \quad (23)$$

Substituting (21) and (22) into equation (1) and the condition (2), we obtain

$$\sum_{k=1}^{+\infty} {}_C D_{0t}^{\alpha} T_k(t)\cdot v_k(x)+\sum_{k=1}^{+\infty} T_k(t)\left[x^{\beta} v_k''(x)\right]''=\sum_{n=1}^{\infty} f_k(t) v_k(x),$$

$$\sum_{k=1}^{+\infty} T_k(0)\cdot v_k(x)=\sum_{k=1}^{\infty}\varphi_k\cdot v_k(x).$$

From the last equalities, by virtue of (4) and completeness of the system of functions $v_k(x)$, $k\in N$ in the space $L_2(0,l)$, it follows the following problem with respect to unknown function $T_k(t)$:

$${}_C D_{0t}^{\alpha} T_k(t)+\lambda_k T_k(t)=f_k(t),\ \ T_k(0)=\varphi_k\ .$$

It is known the solution of this problem represented as follows [ ]

$$T_k(t)=E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k+\int_0^t (t-z)^{\alpha-1} E_{\alpha,\alpha}\left[-\lambda_k (t-z)^{\alpha}\right] f_k(z)\,dz.$$

Substituting this expression of $T_k(t)$ into (21), we obtainformula (20).

Now, we show uniformly convergence of the series

$$u(x,t),\ u_x(x,t),\ x^\beta u_{xx}(x,t),\ \left[x^\beta u_{xx}(x,t)\right]_x,\ {}_C D_{0t}^\alpha u(x,t).$$

Using the rule of integrating by parts we rewrite (20) as follows

$$u(x,t)=\sum_{k=1}^{+\infty}E_{\alpha,1}\left(-\lambda_k t^\alpha\right)\varphi_k v_k(x)+\sum_{k=1}^{+\infty}E_{\alpha,\alpha+1}\left(-\lambda_k t^\alpha\right)f_k(0)v_k(x)+$$

$$+\sum_{k=1}^{+\infty}\int_0^t (t-z)^\alpha E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^\alpha\right]f_k'(z)dz\cdot v_k(x). \tag{24}$$

The following lemma is valid [].

Lemma 4.

We show absolute and uniformly convergence of the series in the right hand side of (24).

From the lemma 4 it follows that Mittag-Leffler function is bounded, i.e.

$$\left|E_{\alpha,\beta}(-z)\right|\le M,\ 0<M<+\infty,\ z>0. \tag{25}$$

One can easily verify that the following inequalities are valid:

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^\alpha\right)\varphi_k\cdot v_k(x)\right|\le M\left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}f_k(0)t^\alpha E_{\alpha,\alpha+1}\left(-\lambda_k t^\alpha\right)v_k(x)\le M\cdot T^\alpha\left[\sum_{k=1}^{+\infty}\lambda_k f_k^2(0)\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\int_0^t (t-z)^\alpha E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^\alpha\right]f_k'(z)dz\cdot v_k(x)\le$$

$$\le M\cdot\sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}}\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k f_k'^2(z)dz\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}.$$

We show the validity of the first inequality. Using (25) and Cauchy-Schwarz inequality, we get

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^\alpha\right)\varphi_k\cdot v_k(x)\right|\le M\sum_{k=1}^{+\infty}\left|\varphi_k\cdot v_k(x)\right|=$$

$$=M\sum_{k=1}^{+\infty}\left|\sqrt{\lambda_k}\varphi_k\cdot\frac{v_k(x)}{\sqrt{\lambda_k}}\right|\le M\left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2\cdot\sum_{k=1}^{+\infty}\frac{v_k^2(x)}{\lambda_k}\right]^{1/2}.$$

Similarly, one can show validity of the second and third inequalities.

From these inequalities, taking into account lemma 1 and lemma 2, it follows that absolutely and uniformly convergence of the series of (24) and $u(x,t)\in C(\overline{\Omega})$.

Differentiating (25) with respect to x, we obtain

$$u_x(x,t)=\sum_{k=1}^{+\infty}E_{\alpha,1}\left(-\lambda_k t^\alpha\right)\varphi_k v_k'(x)+\sum_{k=1}^{+\infty}E_{\alpha,\alpha+1}\left(-\lambda_k t^\alpha\right)f_k(0)v_k'(x)+$$

$$+\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha} E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^{\alpha}\right] f_k'(z)dz\cdot v_k'(x). \qquad (26)$$

Analogously, it can be shown that the following inequalities hold:

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k v_k'(x)\right| \le M\left[\sum_{k=1}^{+\infty}\lambda_k\varphi_k^2\cdot\sum_{k=1}^{+\infty}\frac{v_k'^2}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right) f_k(0) v_k'(x)\right| \le M\cdot T^{\alpha}\left[\sum_{k=1}^{+\infty}\lambda_k f_k^2(0)\cdot\sum_{k=1}^{+\infty}\frac{v_k'^2}{\lambda_k}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha} E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^{\alpha}\right] f_k'(z)dz\cdot v_k'(x) \le$$

$$\le M\cdot\sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}}\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k f_k'^2(z)dz\cdot\sum_{k=1}^{+\infty}\frac{v_k'^2(x)}{\lambda_k}\right]^{1/2}.$$

From the last inequalities by virtue of the conditions of the lemma …it follows that the absolute and uniformly convergence of the series in (26).

Now, differentiating (26) with respect to x and multiplying the both sides of the taken equality to x^{β}, we obtain

$$x^{\beta}u_{xx}(x,t) = \sum_{k=1}^{+\infty}E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k x^{\beta}v_k''(x) + \sum_{k=1}^{+\infty}E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right) f_k(0) x^{\beta}v_k''(x) +$$

$$+\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha} E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^{\alpha}\right] f_k'(z)dz\cdot x^{\beta}v_k''(x). \qquad (27)$$

We show the convergence of the series in (27). Firstly, we show that the following inequalities are valid:

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k x^{\beta}v_k''(x)\right| \le M\left[\sum_{k=1}^{+\infty}\lambda_k^2\varphi_k^2\cdot\sum_{k=1}^{+\infty}\frac{\left[x^{\beta}v_k''(x)\right]^2}{\lambda_k^2}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)t^{\alpha} f_k(0) x^{\beta}v_k''(x)\right| \le M\cdot T^{\alpha}\left[\sum_{k=1}^{+\infty}\lambda_k^2 f_k^2(0)\cdot\sum_{k=1}^{+\infty}\frac{\left[x^{\beta}v_k''(x)\right]^2}{\lambda_k^2}\right]^{1/2},$$

$$\sum_{k=1}^{+\infty}\int_0^t (t-z)^{\alpha} E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^{\alpha}\right] f_k'(z)dz\cdot x^{\beta}v_k''(x) \le$$

$$\le M\cdot\sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}}\left[\int_0^T\sum_{k=1}^{+\infty}\lambda_k^2 f_k'^2(z)dz\cdot\sum_{k=1}^{+\infty}\frac{x^{2\beta}v_k''^2(x)}{\lambda_k^2}\right]^{1/2}.$$

Let’s prove the first inequality. Taking into account (25), we get

$$\sum_{k=1}^{+\infty}\left|E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k x^{\beta}v_k''(x)\right| \le M\sum_{k=1}^{+\infty}\left|\varphi_k x^{\beta}v_k''(x)\right| = M\sum_{k=1}^{+\infty}\left|\lambda_k\varphi_k\cdot\frac{x^{\beta}v_k''(x)}{\lambda_k}\right|.$$

Applying Cauchy-Schwarz inequality from the last, we obtain proof of the first inequality. The second and third inequalities are proved similarly.

Using the same method one can prove absolute and uniformly convergence of the series in $\left[x^{\beta}u_{xx}(x,t)\right]_x$.

Now, we investigate ${}_C D_{0t}^{\alpha} u(x,t)$. Applying differential operator ${}_C D_{0t}^{\alpha}$ to the both sides of (24) and taking into account ${}_C D_{0t}^{\alpha} T_k(t) = -\lambda_k T_k(t)$ and also the view of the function $T_k(t)$, we find

$$
{}_C D_{0t}^{\alpha} u(x,t) = -\sum_{k=1}^{+\infty} \lambda_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k v_k(x) - \sum_{k=1}^{+\infty} \lambda_k E_{\alpha,\alpha+1}\left(-\lambda_k t^{\alpha}\right) f_k(0) v_k(x) -
$$

$$
-\sum_{k=1}^{+\infty} \lambda_k \int_0^t (t-z)^{\alpha} E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^{\alpha}\right] f_k'(z)\,dz \cdot v_k(x) .
$$

And convergence of the series in ${}_C D_{0t}^{\alpha} u(x,t)$ follows by the following inequalities

$$
\sum_{k=1}^{+\infty} \left|\lambda_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\varphi_k v_k(x)\right| \le M\left[\sum_{k=1}^{+\infty} \lambda_k^3 \varphi_k^2 \cdot \sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k}\right]^{1/2},
$$

$$
\sum_{k=1}^{+\infty} \left|\lambda_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right) t^{\alpha} f_k(0) v_k(x)\right| \le M T^{\alpha}\left[\sum_{k=1}^{+\infty} \lambda_k^3 f_k^2(0) \cdot \sum_{k=1}^{+\infty} \frac{v_k^2(x)}{\lambda_k}\right]^{1/2},
$$

$$
\sum_{k=1}^{+\infty} \left|\lambda_k \int_0^t (t-z)^{\alpha} E_{\alpha,\alpha+1}\left[-\lambda_k (t-z)^{\alpha}\right] f_k'(z)\,dz \cdot v_k(x)\right| \le
$$

$$
\le M \cdot \sqrt{\frac{T^{2\alpha+1}}{2\alpha+1}} \left[\int_0^T \sum_{k=1}^{+\infty} \lambda_k^3 f_k'^2(z)\,dz \cdot \sum_{k=1}^{+\infty} \frac{v^2(x)}{\lambda_k}\right]^{1/2}.
$$

These inequalities are proved the same method as above-proved inequalities. From these inequalities, it follows the absolute and uniform convergence of the series in ${}_C D_{0t}^{\alpha} u(x,t)$.

Theorem 1 has been proved.

VI. THE UNIQUENESS OF THE SOLUTION OF THE PROBLEM

Now, we prove the uniqueness of the solution of the problem. For this aim, we introduce the following function

$$
T_k(t) = \int_0^l u(x,t) v_k(x)\,dx . \tag{28}
$$

Based on (27), we consider the following auxiliary function

$$
T_{k,\varepsilon}(t) = \int_{\varepsilon}^{l-\varepsilon} u(x,t) v_k(x)\,dx , \tag{29}
$$

where ε is sufficiently small positive number.

Applying differential operator ${}_C D_{0t}^{\alpha}$ to (29) and using homogeneous equation corresponding equation (1), we get

$$ {}_C D_{0t}^{\alpha} T_{k,\varepsilon}(t) = -\int_{\varepsilon}^{l-\varepsilon} \left[x^{\beta} u_{xx}(x,t) \right]_{xx} \cdot v_k(x)\, dx . $$

Using the rule of integration by parts four times from the last equality, we get

$$ {}_C D_{0t}^{\alpha} T_{k,\varepsilon}(t) = -\Big\{ v_k(x) \left[x^{\beta} u_{xx}(x,t) \right]_x - v_k'(x) x^{\beta} u_{xx}(x,t) + x^{\beta} v_k''(x) u_x(x,t) - $$

$$ - \left[x^{\beta} v_k''(x) \right]' u(x,t) \Big\} \Big|_{x=\varepsilon}^{x=l-\varepsilon} - \int_{\varepsilon}^{l-\varepsilon} \left[x^{\beta} v_k''(x) \right]'' u(x,t)\, dx . \qquad (30) $$

Passing to the limit as $\varepsilon \to 0$ and taking (4) and (28) into account, from (30), we derive

$$ {}_C D_{0t}^{\alpha} T_k(t) + \lambda_k T_k(t) = 0 . \qquad (31) $$

From (28), we have

$$ T_k(0) = \int_0^l u(x,0) v_k(x)\, dx = \varphi_k . \qquad (32) $$

It is known that, the solution of the equation (31) satisfying condition (32) is represented as follows

$$ T_k(t) = \varphi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha} \right). $$

Let $\varphi(x) \equiv 0$, $x \in [0,l]$. Then from the last equality taking into account (28) for all $t \in [0,\mathrm{T}]$ and $k \in N$ it follows that

$$ \int_0^l u(x,t) v_k(x)\, dx = 0 . \qquad (33) $$

As since problem {(4),(5)} is self-adjoint, its eigenfunctions will be complete system in $L_2[0,l]$. Taking this account from (33), we get $u(x,t) \equiv 0$ almost everywhere on $[0,l]$ for all $t \in [0,T]$. By virtue of $u(x,t) \in C(\bar{\Omega})$, we obtain $u(x,t) \equiv 0$ in $\bar{\Omega}$. Thus, homogeneous problem has only trivial solution and this give us the uniqueness of the solution of the considered problem.

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