O‘ZBEKISTON RESPUBLIKASI

OLIY TA’LIM, FAN VA INNOVATSIYALAR VAZIRLIGI

FARG‘ONA DAVLAT UNIVERSITETI

FarDU. ILMIY XABARLAR

1995-yildan nashr etiladi
Yilda 6 marta chiqadi

2024/3-SON
ILOVA TO‘PLAM

НАУЧНЫЙ ВЕСТНИК. ФерГУ

Издаётся с 1995 года
Выходит 6 раз в год

M U N D A R I J A

Aniq va tabiiy fanlar

MATEMATIKA

FIZIKA-TEXNIKA

FarDU. Ilmiy xabarlar – Scientific journal of the Fergana State University

Volume 30 Issue 3, 2024-yil DOI: 10.56292/SJFSU/vol30_iss3_2t/a334

UO‘K: 517.95

KASR TARTIBLI HILFER DIFFERENSIAL OPERATORINI O‘Z ICHIGA OLUVCHI BENJAMIN-BONA-MAHONI TENGLAMASI UCHUN CHEGARAVIY MASALA YECHIMINING MAVJUDLIGI HAQIDA

О СУЩЕСТВОВАНИИ РЕШЕНИЯ КРАЕВОЙ ЗАДАЧИ ДЛЯ УРАВНЕНИЯ БЕНДЖАМИНА-БОНА-МАХОНИ, ВКЛЮЧАЮЩЕГО ДИФФЕРЕНЦИАЛЬНЫЙ ОПЕРАТОР ХИЛЪФЕРА ДРОБНОГО ПОРЯДКА

THE EXISTENCE OF THE SOLUTION OF A BOUNDARY VALUE PROBLEM FOR THE BENJAMIN, BONA AND MAHONY EQUATION INCLUDING THE HILFER FRACTIONAL DIFFERENTIAL OPERATOR

Mamanazarov Azizbek Otajon ugli[1]
[1]Associate Professor of Fergana State University **ORCID ID 0000-0001-8729-7852**

Djuraeva Yokutkhon Bakhromjon qizi[2]
[2]Master Student of Fergana State University **ORCID ID 0009-0000-7588-0370**

Annotatsiya

Mazkur maqolada kasr tartibli Hilfer operatorini o‘z ichiga oluvchi Benjamin-Bona-Mahony tenglamasi uchun boshlang‘ich chegaraviy masala yechiming mavjud bo‘lmasligi shartlari aniqlangan.

Аннотация

В данной статье определяются условия отсутствия решения начально-краевой задачи для уравнения Бенджамина-Бона-Махони, содержащего оператор Хильфера дробного порядка.

Abstract

In this article, the conditions for the non-existence of the solution to the initial boundary value problem for Benjamin-Bona-Mahony equation containing fractional order Hilfer operators are determined.

Kalit so‘zlar: *kasr tartibli Hilfer differensial operatori, Benjamin-Bona-Mahony tenglamasi, yechimning mavjudligi.*

Ключевые слова: *дифференциальный оператор Хильфера дробного порядка, уравнение Бенджамина-Бона, Махони, существование решения.*

Key words: *fractional Hilfer differential operator, Benjamin, Bona and Mahony equation, the existence of the solution.*

INTRODUCTION

Benjamin, Bona and Mahony proposed the following equation

$$u_t\left(t,x\right)-u_{txx}\left(t,x\right)+u(t,x)u_x\left(t,x\right)=0$$

to describe long waves on the water, where $u\left(t,x\right)$ is the velocity of the fluid, t is time, and x is position. The Benjamin, Bona and Mahony (BBM) equation is a nonlinear equation that is used in many fields such as fluid mechanics, traffic flow, and mathematical physics (see [1]).

The BBM equation provides a mathematical model for the propagation of long waves in bodies of water, such as oceans, lakes, and rivers. It helps in understanding the behavior of waves under different conditions, including dispersion effects.

Moreover, the BBM equation can shed light on phenomena like wave breaking and turbulence in water waves. Understanding these processes is crucial for various applications, including coastal engineering and offshore structure design.

The BBM equation is of interest in the field of nonlinear PDEs and applied mathematics. Studying its properties, such as existence and uniqueness of solutions, stability, and long-term behavior, contributes to the broader understanding of nonlinear wave equations.

Fractional calculus pertains to the branch of mathematical analysis focused on exploring and utilizing integrals and derivatives of non-integer orders. Over the past few decades, it has gained notable importance owing to its wide-ranging applications across numerous scientific and engineering domains. These include but are not limited to areas such as viscoelasticity, signal processing, electromagnetics, fluid mechanics, and optics.

The aim of the work is to consider the time fractional analogue of the BMM equation and to study under which conditions the solution of the boundary value problem for the equation of Benjamin, Bona and Mahony does not exist.

2. Preliminaries

In this section, we recall the definitions and properties of fractional order integral and differential operators, which will be used throughout the paper.

For detailed information we refer to the reader the references [2],[3],[4].

Definition 2.1. (Riemann-Liouville integral). The Riemann-Liuville fractional integral of order $\alpha > 0$ with lower limit a is defined for locally integrable functions f: $[a, b] \to$ R as

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds. \qquad (2.1)$$

where $\Gamma(\alpha)$ is Euler's gamma function.

The Riemann-Liuville fractional integral of order $\alpha > 0$ with upper limit b is defined as

$$I_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds. \qquad (2.2)$$

Definition 2.2. The Riemann-Liouville left-hand fractional derivate $D_{a+}^{\alpha} f$ of $\alpha(0<\alpha<1)$ order is defined as

$$D_{a+}^{\alpha} f(t) = \frac{d}{dt} I_{a+}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) ds. \qquad (2.3)$$

Definition 2.3. The Riemann-Liouville right-hand fractional derivative $D_{b-}^{\alpha} f$ of $\alpha(0<\alpha<1)$ order is defined as

$$D_{b-}^{\alpha} f(t) = -\frac{d}{dt} I_{b-}^{1-\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (s-t)^{-\alpha} f(s) ds. \qquad (2.4)$$

Definition 2.4. Caputo's fractional derivative of $\alpha(n-1<\alpha<n)$ fractional order is defined by

$${}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau. \qquad (2.5)$$

Definition 2.5. The Hilfer derivative $D_{a+}^{\alpha,\beta} f$ of order $0<\alpha<1$ and type $0 \le \beta \le 1$ is defined by

$$D_{a+}^{\alpha,\beta} f(t) = I_{a+}^{\beta(1-\alpha)} \frac{d}{dt} I_{a+}^{(1-\beta)(1-\alpha)} f(t). \qquad (2.6)$$

where I_{a+}^{σ}, $\sigma > 0$ is the Riemann-Liouville fractional integral operator.

We note that Hilfer fractional derivative becomes the Riemann-Liouville derivative when $\beta = 0$ and the Caputo fractional derivative when $\beta = 1$ (see [4]).

Lemma 2.1 Let $\alpha>0,\ p\ge 1,\ q\ge 1$ and $\frac{1}{p}+\frac{1}{q}\le 1+\alpha$ ($\frac{1}{p}+\frac{1}{q}=1+\alpha$ in case $p\ne 1$ and $q\ne 1$). Also, let $\varphi\in L_p(a,b)$ and $\psi\in L_q(a,b)$, then

$$\int_a^b \varphi(t) I_{a+}^{\alpha}\psi(t)dt=\int_a^b \psi(t) I_{b-}^{a}\varphi(t)dt. \qquad (2.7)$$

3. The non-existence of the solution of the time-fractional Benjamin, Bona and Mahony equation

Let us denote by $\Pi_{a,b}$ a rectangular domain of $\mathbf{R}^2$, i.e.

$$\Pi_{a,b}=\left\{(t,x)\in\mathbf{R}^2: 0<t<T, a<x<b\right\}.$$

In the domain $\Pi_{a,b}$, we consider the following time-fractional Benjamin, Bona and Mahony equation

$$D_{0+,t}^{\alpha,\beta}u(t,x)-u_{txx}(t,x)+u(t,x)u_x(t,x)=0 \qquad (3.1)$$

with the following initial condition

$$I_{0+,t}^{\gamma-1}u(0,x)=u_0(x),\quad x\in[a,b], \qquad (3.2)$$

where $D_{a+}^{\alpha,\beta}$ is the Hilfer derivative of order $0<\alpha<1$ and type $0\le\beta\le 1$, $u_0(x)$ is a given function, γ is a real number such that $\gamma=(1-\beta)(1-\alpha)+1$.

We should note that from the equation (3.1) when $\alpha=1$, $\beta=0$ it follows the equation (1.1).

We denote by $\Phi(\Pi_{a,b})$ the class of test functions $\varphi(t,x)$ defined in the domain $\Phi(\Pi_{a,b})=\{(t,x)\in R^2:\ 0<t<T;\ a<x<b\}$ with arbitrary parameters $T>0,\ a,b\in R,\ a<b$ and possessing the following properties:

$$(i)\ \varphi_t,\varphi_{xx}\in C(\Pi_{a,b});$$

$$(ii)\ \varphi_x\ge 0\ in\ \Pi_{a,b};$$

$$(iii)\ I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)=0\ in\ x\in(a,b)\ and\ t=T;$$

$$(iv)\ \zeta(\Pi_{a,b})=\iint_{\Pi_{a,b}}\frac{(L^*\varphi)^2}{\varphi_x}dtdx<+\infty,$$

where $L^*\varphi=-I_{T-,t}^{(1-\beta)(1-\alpha)}\frac{d}{dt}I_{T-,t}^{\beta(1-\alpha)}\varphi+\varphi_{xxt}$.

Suppose that there exists a smooth solution $u_{xx}, D_{0+,t}^{\alpha,\beta}u\in C([a,b]\times[0,t])$ of the problem (3.1) - (3.2) and the number $T>0$.

Multiplying the equation (3.1) by an arbitrary function $\varphi\in\Phi(\Pi_{a,b})$ and then integrating over $\Pi_{a,b}$, we obtain

$$\iint_{\Pi_{a,b}}\varphi(t,x)D_{0+,t}^{\alpha,\beta}u(t,x)dtdx-\iint_{\Pi_{a,b}}\varphi(t,x)u_{txx}(t,x)dtdx+$$

$$+\iint_{\Pi_{a,b}}\varphi(t,x)u(t,x)u_{txx}(t,x)dtdx=0. \qquad (3.3)$$

Hence, applying the rule of integration by parts, it is easy to see that

$$\iint_{\Pi_{a,b}} \varphi(t,x)u_{txx}(t,x)dtdx =$$

$$=\int_0^T \left[u_{tx}(t,x)\varphi(t,x)-u_t\varphi_x(t,x)\right]\Big|_a^b dt-\int_a^b u(t,x)\varphi_{xx}(t,x)\Big|_0^T dx+$$

$$+\iint_{\Pi_{a,b}} \varphi_{xxt}(t,x)u(t,x)dtdx, \qquad (3.4)$$

$$\iint_{\Pi_{a,b}} \varphi(t,x)u(t,x)u_x(t,x)dtdx =$$

$$=\frac{1}{2}\int_0^T \varphi(t,x)u^2(t,x)\Big|_a^b dt-\frac{1}{2}\iint_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx. \qquad (3.5)$$

Moreover, considering Definition 2.5 and applying Lemma 2.1, we get

$$\iint_{\Pi_{a,b}} \varphi(t,x)D_{0+,t}^{\alpha,\beta}u(t,x)dtdx=\iint_{\Pi_{a,b}} \varphi(t,x)I_{0+}^{\beta(1-\alpha)}\frac{d}{dt}I_{0+}^{(1-\beta)(1-\alpha)}u(t,x)dtdx=$$

$$=\iint_{\Pi_{a,b}} I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)\frac{d}{dt}I_{0+}^{(1-\beta)(1-\alpha)}u(t,x)dtdx\,.$$

By applying the rule of integration by parts to the last integral and then using Lemma 2.1, we obtain

$$\iint_{\Pi_{a,b}} \varphi(t,x)D_{0+,t}^{\alpha,\beta}u(t,x)dtdx=\int_a^b\left\{I_{0+,t}^{(1-\beta)(1-\alpha)}u(t,x)I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)\right\}\Big|_0^T dx-$$

$$-\iint_{\Pi_{a,b}} \frac{d}{dt}I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)I_{0+}^{(1-\beta)(1-\alpha)}u(t,x)dtdx=$$

$$\int_a^b\left\{I_{0+,t}^{(1-\beta)(1-\alpha)}u(t,x)I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)\right\}\Big|_0^T dx-$$

$$-\iint_{\Pi_{a,b}} u(t,x)I_{T-,t}^{(1-\beta)(1-\alpha)}\frac{d}{dt}I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)dtdx. \qquad (3.6)$$

Taking (3.4), (3.5), (3.6) and Definition 2.3 into account from (3.3), we have

$$\frac{1}{2}\iint_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx=\iint_{\Pi_{a,b}} u(t,x)\left(L^*\varphi\right)(t,x)dtdx+$$

$$+\int_a^b\left\{I_{0+,t}^{(1-\beta)(1-\alpha)}u(t,x)I_{T-,t}^{\beta(1-\alpha)}\varphi(t,x)\right\}\Big|_0^T dx+\int_0^T B\left(u(t,x),\varphi(t,x)\right)\Big|_a^b dt\,, \qquad (3.7)$$

where $B\left(u(t,x),\varphi(t,x)\right)=\frac{1}{2}u^2(t,x)\varphi(t,x)-u_t(t,x)\varphi_x(t,x)-u_{tx}(t,x)\varphi(t,x).$

From (3.7), based on the property *(iii)* of the function $\varphi(t,x)$ and taking (3.2) into account, we have

$$\frac{1}{2}\iint\limits_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx = \iint\limits_{\Pi_{a,b}} u(t,x)(L^*\varphi)(t,x)dtdx + \int_0^T B(u(t,x),\varphi(t,x))\Big|_a^b dt -$$

$$-\int_a^b u_0(x) I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)dx. \tag{3.8}$$

Applying Holder's and Young's inequalities to the right-hand side of (3.8), we obtain the following estimate

$$\left|\iint\limits_{\Pi_{a,b}} u(t,x)(L^*\varphi)(t,x)dtdx\right| = \left|\iint\limits_{\Pi_{a,b}} u(t,x)\sqrt{\varphi_x(x,t)}\frac{(L^*\varphi)(t,x)}{\sqrt{\varphi_x(x,t)}}dtdx\right| \le$$

$$\left(\iint\limits_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx\right)^{1/2}\left(\iint\limits_{\Pi_{a,b}} \frac{\left((L^*\varphi)(t,x)\right)^2}{\varphi_x(x,t)}dtdx\right)^{1/2} \le$$

$$\frac{1}{2}\iint\limits_{\Pi_{a,b}} u^2(t,x)\varphi_x(t,x)dtdx + \frac{1}{2}\iint\limits_{\Pi_{a,b}} \frac{\left((L^*\varphi)(t,x)\right)^2}{\varphi_x(x,t)}dtdx.$$

Taking this relation into account from (3.8), we come

$$0 \le \frac{1}{2}\zeta(\Pi_{a,b}) + \int_0^T B(u(t,x),\varphi(t,x))\Big|_a^b dt - \int_a^b u_0(x) I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\Big|_{t=0}dx. \tag{3.9}$$

Based on the inequality (3.9), we have the following assertion:

Theorem 3.1. Assume that $u_0(x) \in L[a,b]$ and there exists a test function $\varphi(x,t) \in \Phi(\Pi_{a,b})$ for which the inclusion

$$B(u(t,x),\varphi(t,x))\Big|_a^b \in L[0,T]$$

holds and the following

$$\frac{1}{2}\zeta(\Pi_{a,b}) + \int_0^T B(u(t,x),\varphi(t,x))\Big|_a^b dt - \int_a^b u_0(x) I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\Big|_{t=0}dx < 0 \tag{3.10}$$

inequality is satisfied. Then, there does not exist a global-in-time solution to the problem {(3.1), (3.2)}.

Proof. Let us assume the opposite i.e. the problem {(3.1), (3.2)} admits a global-in-time solution in $\Pi_{a,b}$. Then, we arrived at a contradiction by inequalities (3.9) and (3.10).

As an example in a rectangular domain $\Pi_{a,b} = \{(t,x) \in \mathbf{R}^2 : 0 < t < T, 0 < x < 1\}$ we consider the time-fractional Benjamin, Bona, Mahony equation (3.1) with the initial condition (3.2) and the following

$$u(t,0) = \tau_1(t),\ u_t(t,0) = \tau_2(t),\ u_{tx}(t,0) = \tau_3(t),\ 0 < t < T \tag{3.11}$$

boundary conditions, where $\tau_1(t)$, $\tau_2(t)$ and $\tau_3(t)$ are given functions, such that $\tau_1, \tau_2, \tau_3 \in L[0,T]$.

Multiplying the equation (3.1) by $\varphi \in \Phi(\Pi_{a,b})$ and doing some computations, we obtain

$$0<\frac{1}{2}\zeta\left(\Pi_{0,1}\right)+\int_0^T B\left(u(t,x),\varphi(t,x)\right)\Big|_a^b dt-\int_0^1 u_0(x)I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\big|_{t=0}dx\,.$$

Assume that $\varphi(t,x)$ satisfies the following conditions

$$\varphi(t,1)=0,\ \varphi_x(t,1)=0,\ 0<t<T. \qquad (3.12)$$

Then, we have

$$B(u,\varphi)\Big|_a^b=-\left[\frac{1}{2}\tau_1^2(t)-\tau_3(t)\right]\varphi(t,0)-\tau_2(t)\varphi_x(t,0).$$

For this problem the following assertion is true:

Theorem 3.2. Let $u_0(x)\in L[0,1]$ and let u be a solution of the problem {(3.1), (3.2), (3.11)} such that $u_{xx}, D_{0+,t}^{\alpha,\beta}u\in C\left([a,b]\times[0,t]\right)$ and let $\varphi\in\Phi\left(\Pi_{0,1}\right)$ be a function satisfying (3.12).

If the following inequality

$$\frac{1}{2}\zeta\left(\Pi_{0,1}\right)<\int_0^T\left[\frac{1}{2}\tau_1^2(t)\varphi(t,0)-\tau_3(t)\varphi(t,0)+\tau_1(t)\varphi_x(t,0)\right]dt+$$

$$+\int_0^1 u_0(x)I_{T-,t}^{\beta(1-\alpha)}\varphi(x,t)\big|_{t=0}dx \qquad (3.13)$$

is valid, then the global-in time solution of the problem {(3.1), (3.2), (3.11)} doesn't exist in $\Pi_{0,1}$.

Example 3.1. If we choose the test function as

$$\varphi(t,x)=(T-t)^2(x-1)^3, \qquad (3.14)$$

then it can be easily shown that the conditions *(i) - (iv)* are fulfilled for the function $\varphi(t,x)$. Also using this function, we can easily obtain

$$\zeta\left(\Pi_{0,1}\right)=\frac{4k_1^2}{3}\frac{T^{3-2\alpha}}{3-2\alpha}+16k_1\frac{T^{2-\alpha}}{2-\alpha}+48T, \qquad (3.15)$$

where $k_1=\Gamma^{-1}(3-\alpha)$.

Then, according to the Theorem 3.2, we have the following assertion:

Corollary 3.1. Let $u_0\in L[0,1]$, $\tau_1,\tau_2,\tau_3\in L[0,T]$. If the following inequality

$$k_2\int_0^1 u_0(x)(x-1)^3(T-t)^{\beta(1-\alpha)+2}dx>\int_0^T\left[\frac{1}{2}\tau_1^2(t)-\tau_3(t)-3\tau_1(t)\right](T-t)^2dt+\frac{2k_1^2}{3}\frac{T^{3-2\alpha}}{3-2\alpha}+$$

$$+8k_1\frac{T^{2-\alpha}}{2-\alpha}+24T$$

holds, then there does not exist a global-in-time solution to the initial-boundary problem {(3.1), (3.2), (3.11)}, where $k_2=-2\Gamma^{-1}\left(\beta(1-\alpha)+3\right)$.

BIBLIOGRAPHY

1. T. B. Benjamin, J. L. Bona, J. J. Mahony. (1972). Model equations for long waves in nonlinear dispersive systems. Phil. Trans. R. Soc. Ser. A. 272, 47–78.

2. R. Hilfer. (2000). Applications of Fractional Calculus in Physics, World Scientific, Singapore, p.87 and p.429.

3. R. Hilfer. (2002). Experimental evidence for fractional time evolution in glass materials, Chem. Physics 284, 399-408.

4. A. A. Kilbas, H. M. Srivastava va J. J. Trujillo. (2006). *Theory and Applications of Fractional Differential Equations,* Elsevier, North-Holland.