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СУБ-ДИФФУЗИЯ ВА ТЎЛҚИН ТЕНГЛАМАЛАРИДАН ИБОРАТ АРАЛАШ ТИПДАГИ ТЕНГЛАМА УЧУН ТРИКОМИ ТИПИДАГИ МАСАЛА

TRICOMI TYPE PROBLEM FOR MIXED TYPE EQUATION WITH SUB-DIFFUSION AND WAVE EQUATION

ЗАДАЧА ТИПА ТРИКОМИ ДЛЯ УРАВНЕНИЯ СМЕШАННОГО ТИПА С УРАВНЕНИЕМ СУБ-ДИФФУЗИИ И ВОЛНОВЫМ УРАВНЕНИЕМ

E.Karimov, S.Kerbal

Аннотация

Мақолада икки тартибли Хилфер ҳосилали суб-диффузия ва классик тўлқин тенгламасида иборат аралаш типдаги тенглама учун Трикоми типидаги масала аралаш соҳада ўрганилган. Асосий қўлланилган усуллар интеграл тенгламалар ва энергия интеграллари усулларидир.

Annotation

In this paper, we have considered a Tricomi type problem for mixed type equation with Hilfer's double order derivative sub-diffusion equation and classical wave equation in a composite domain. Main methods of the investigation are a method of integral equations and energy integrals' method.

Аннотация

В статье рассмотрена задача типа Трикоми для уравнения смешанного типа с уравнением суб-диффузии и волновым уравнением. Основными методами являются метод интегральных уравнений и метод интегралов энергии.

Таянч сўз ва иборалар: *Трикоми масаласи, икки тартибли Хилфер ҳосиласи, интеграл тенгламалар усули.*

Keywords and expressions: *Tricomi problem, double order Hilfer's derivative, method of integral equations*

Ключевые слова и выражения: *задача Трикоми, производное Хилфера с двумя порядками, метод интегральных уравнений.*

Introduction and formulation of a problem.

Due to both practical [1] and theoretical importance [2], Fractional Calculus is developing intensively.

Solutions of differential equations with the most popular fractional derivatives such as the Riemann-Liouville, Caputo and Hilfer's derivatives were subject of investigations in [3], [4], [5]. Analyzing a huge amount of works devoted to the study of direct and inverse problems for partial differential equations (PDEs) with fractional order operators, we noted the only study closely related to the present topic. Several boundary value problems (BVPs) for mixed type equations with Riemann-Liouville fractional differential operator (FDO) were studied for unique solvability in works [6],[7]. BVP with integral form conjugation conditions for PDEs with both Riemann-Liouville and Caputo FDOs were subject of series of investigations [8], [9], [10]. In these works, authors used an explicit solution of certain BVP for fractional diffusion equation studied by A.Pskhu [5].

For the first time, generalized the Riemann-Liouville FDO, which is alternatively named as Hilfer FDO introduced by Hilfer [11]. The Cauchy and some BVPs for ODEs and PDEs with Hilfer FDO investigated by many authors, for instance [4], [12].

In this paper, we are aimed to study BVP with integral form conjugation condition in a mixed domain consisted of characteristic triangle and rectangle, for a mixed type PDE with diffusion equation involving Hilfer's double order FDO, which is introduced in [13]. In recent work [14], Tricomi type problem with integral conjugation condition for mixed type equation with Hilfer's derivative was investigated.

Consider the following mixed type equation

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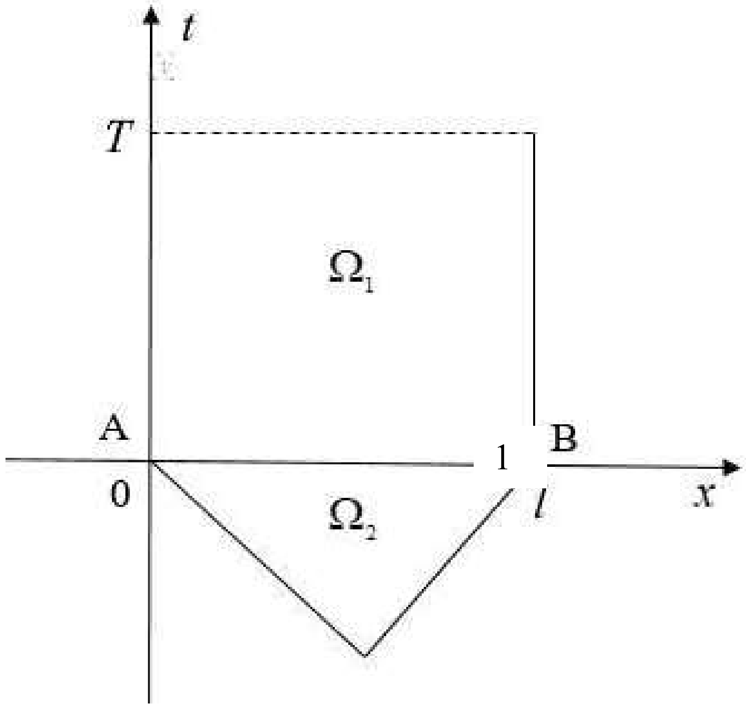

$$0=\begin{cases} D_{0t}^{(\alpha,\beta)\mu}u(x,t)-u_{xx}(x,t), & t>0, \\ u_{tt}(x,t)-u_{xx}(x,t), & t<0 \end{cases} \tag{1}$$

in a composite domain $\Omega=\Omega_1\cup\Omega_2\cup AB$. Here

$$D_{0t}^{(\alpha,\beta)\mu}f = I_{0t}^{\mu(1-\alpha)}\frac{d}{dt}I_{0t}^{(1-\mu)(1-\beta)}f \quad \left(0<\alpha,\beta\le 1;\ 0\le\mu\le 1\right)$$

is the Hilfer's double-order fractional derivative of orders α and β of type μ [13],

$I_{0t}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int\limits_0^t\frac{f(z)dz}{(t-z)^{1-\alpha}}$ is the Riemann-Liouville fractional integral of order α [2].

Problem. To find a function $u(x,t)$, which is continuous in $\overline{\Omega}\setminus AB$, its Hilfer's double-order derivative is continuous in Ω_1 and it has continuous second order partial derivatives in Ω_2, and it satisfies Eq. (1) in Ω together with boundary conditions

$$u(0,t)=u(1,t)=0,\quad 0\le t\le T, \tag{2}$$

$$u(x/2,-x/2)=\psi(x),\ 0\le x\le 1, \tag{3}$$

conjugation conditions on AB

$$\lim_{t\to+0}t^{(1-\mu)(1-\beta)}u(x,t)=u(x,-0),\quad 0\le x\le 1, \tag{4}$$

$$\lim_{t\to+0}t^{1-\alpha}\left(t^{(1-\mu)(1-\beta)}u(x,t)\right)_t=u_t(x,-0),\quad 0<x<1. \tag{5}$$

Here $\psi(x)$ is a given function such that $\psi(0)=0$.

1. Main functional relations.

Introduce a notation

$$\tau_1(x) = \lim_{t\to+0} t^{(1-\mu)(1-\beta)} u(x,t), \quad 0 \le x \le 1. \tag{6}$$

Solution of (1) in Ω_1 satisfying condition (2) and (6) has a form [13]

$$u(x,t) = \int_0^1 \tau_1(\xi) M(x,\xi;t)\, d\xi, \tag{7}$$

where

$$M(x,\xi;t) = 2t^{(1-\mu)(\beta-1)}\Gamma(\alpha+\mu(1-\beta)) \sum_{k=1}^{\infty} E_{\beta+\mu(\alpha-\beta),\beta+\mu(1-\beta)}\left(-(k\pi)^2 t^{\beta+\mu(\alpha-\beta)}\right) \sin k\pi x \sin k\pi\xi, \tag{8}$$

$E_{\alpha,\beta}(z) = \sum\limits_{n=0}^{\infty} \dfrac{z^n}{\Gamma(\alpha n+\beta)}$ is a two-parameter Mittag-Leffler function[2].

Doing the same as in work [14], we deduce

$$\nu_1(x) = \frac{(\beta+\mu(\alpha-\beta))\Gamma(\alpha+\mu(1-\beta))}{\Gamma(\beta+\mu(1-\beta))} \tau_1''(x), \quad 0 < x < 1, \tag{9}$$

where $\nu_1(x) = \lim\limits_{t\to+0} t^{1-\alpha}\left(t^{(1-\mu)(1-\beta)} u(x,t)\right)_t, \quad 0 < x < 1.$

Solution of (1) in Ω_2 satisfying initial conditions

$$u(x,-0) = \tau_2(x), \ 0 \le x \le 1, \quad u_t(x,-0) = \nu_2(x), \quad 0 < x < 1 \tag{10}$$

can be written by D'Halembert's formula

$$u(x,t) = \frac{1}{2}\left[\tau_2(x-t) + \tau_2(x+t) + \int_{x-t}^{x+t} \nu_2(z)\,dz\right]. \tag{11}$$

Substituting (11) into (3) we deduce

$$\nu_2(x) = \tau_2'(x) - 2\psi'(x), \ 0 < x < 1. \tag{12}$$

Relations (9) and (12) are main functional relations, which are essential in further.

2. A unique solvability of the problem.

Based on conjugation conditions (4), (5) from main functional relations (9) and (12) yields

$$\tau_1''(x) - A\tau_1'(x) = 2A\psi'(x), \quad 0 < x < 1, \tag{13}$$

where $A = \dfrac{\Gamma(\beta+\mu(1-\beta))}{(\beta+\mu(\alpha-\beta))\Gamma(\alpha+\mu(1-\beta))}$. Boundary conditions in (2) yield

$$\tau_1(0) = \tau_1(1) = 0. \tag{14}$$

Problem (13)-(14) has an explicit solution in a form of

$$\tau_1(x) = 2\int_0^1 \psi'(\xi) G(x,\xi)\, d\xi \tag{15}$$

with

$$G(x,\xi)=\frac{1}{e^{Ax}-e^{A(x-1)}}\begin{cases}\left(1-e^{A\xi}\right)\left(1-e^{A(x-1)}\right),\ 0\le\xi\le x,\\ \left(1-e^{A(\xi-1)}\right)\left(1-e^{Ax}\right),\ x\le\xi\le 1.\end{cases} \tag{16}$$

Using representation (15), unknown function $\nu_1(x)$ will be found by formula (9). Due to conditions (4), (5), other two unknown functions $\tau_2(x),\nu_2(x)$ will be determined explicitly. The solution of the considered problem will be recovered by formulas (7) and (11) in domains Ω_1 and Ω_2, respectively.

A uniqueness of the solution of the problem can be proved in a similar way as it is done in [14]. Namely, we multiply (9) by function $\tau_1(x)$ and integrate along AB, then easily will get

$$A\int_0^1\tau_1(x)\nu_1(x)dx+\int_0^1\left(\tau_1'(x)\right)^2dx=0. \tag{17}$$

Considering (4), (5) and (12) one can readily prove that $\int_0^1\tau_1(x)\nu_1(x)dx\ge 0$. Since $\beta+\mu(\alpha-\beta)>0$, then from (17) it follows that $\tau_1(x)=0$. Further, considering solution of the first BVP for Eq.(1) in Ω_1, we will get $u(x,t)=0$ in Ω_1. Due to (4), one can easily deduce that $u(x,t)=0$ in Ω.

Finally, we are able now to formulate our result as the following

Theorem. If $\psi(x)\in C[0,1]\cap C^1(0,1)$, then formulated problem has a unique solution represented as follows

$$u(x,t)=2\theta(t)\int_0^1\int_0^1\psi'(\eta)G(\xi,\eta)M(x,\xi;t)d\xi d\eta+$$

$$+\theta(-t)\left[\int_0^1\psi'(\eta)\left[G(x-t,\eta)+G(x+t,\eta)\right]d\eta+\frac{1}{A}\int_{x-t}^{x+t}dz\int_0^1\psi'(\eta)G_{zz}(z,\eta)d\eta\right],$$

where $\theta(t)=1$ for $t\ge 0$ and $\theta(t)=0$ for $t<0$.

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