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VAZN FUNKSIYASIGA EGA BOʻLGAN RIMAN-LIUVIL VA ATANGANA-BALEANU KASR TARTIBLI OPERATORLAR QATNASHGAN TOʻLQIN TENGLAMASI UCHUN CHEGARAVIY MASALA

ON A BOUNDARY VALUE PROBLEM FOR A TIME-FRACTIONAL WAVE EQUATION WITH THE WEIGHTED RIEMANN-LIOUVILLE AND ATANGANA-BALEANU DERIVATIVES

ГРАНИЧНАЯ ЗАДАЧА ДЛЯ УРАВНЕНИЯ КОЛЕБАНИЯ, С УЧАСТИЕМ ДРОБНОГО ОПЕРАТОРА РИМАНА-ЛИУВИЛЛЯ И АТАНГАНА-БАЛЕАНУ, ИМЕЮЩЕГО ВЕСОВУЮ ФУНКЦИЮ

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Annotatsiya

Ushbu maqolada vazn funksiyasiga ega boʻlgan Riman-Liuvil va Atangana-Baleanu kasr tartibli operatorlar qatnashgan toʻlqin tenglamasi uchun bir chegaraviy masala bayon qilingan va oʻrganilgan. Masala Furye usuli yordamida tadqiq qilingan. Bunda vaqt oʻzgaruvchisi boʻyicha hosi lqilingan masala yechimi Laplas almashtirishi yordamida topilgan. Qoʻyilgan masalaning yechimi qator koʻrinishida topilgan.

Аннотация

В данной статье была исследована и изучена одна граничная задача для уравнения колебания, с участием дробного оператора Римана-Лиувилля и Атангана-Балеану, имеющий функцию с весом. Задача была решена методом Фурье. При нахождении решения задачи, использована преобразование Лапласа относительно временной переменной. Решение поставленной задачи было найдено в виде ряда.

Abstract

In the article a boundary value problem has been formulated and studied for a wave equation involving Riemann-Liuville and Atangana-Baleanu fractional operators with weight functions. The problem was investigated using the Fourier method. In this case, a solution of the problem with respect to the time variable was found using the Laplace transform. The solution to the problem has been represented by series.

Kalit soʻzlar: *kasr tartibli differensial operator, vazn funksiyasiga ega boʻlgan Atangana-Baleanu kast tartibli operator, differensial tenglama, Kosh imasalasi.*

Ключевые слова: *дифференциальный оператор дробного порядка, дифференциальный оператор Атангана-Балеану дробного порядка с весовой функцией, дифференциальное уравнение, задача Коши.*

Key words: *the fractional differential operator, the weighted Atangana-Baleanu fractional derivative, differential equation, Cauchy problem.*

KIRISH

Ushbu maqolada quyidagi operatorlar bilan tanishamiz:

$f(t)\in W^1(0,T]$ funksiyaning $0<\beta<1$ tartibli $w(t)$ vaznli Kaputo ma'nosidagi Atangana-Baleanu kasr tartibli differensial operatori quyidagicha aniqlanadi [1]:

$$\left({}_cD_w^\beta f\right)(t)=\frac{M(\beta)}{1-\beta}\frac{1}{w(t)}\int_0^t E_\beta\left[-\frac{\beta}{1-\beta}(t-s)^\beta\right]\frac{d}{ds}(wf)(s)ds,\qquad t>0. \tag{1}$$

Bu yerda $w\in C^1(0,T]$, $w,w'>0$ da $(0,T]$, $M(\alpha)$ shunday normallashtirish funksiyasiki, $M(0)=M(1)=1$. $E_\alpha(z)=\sum\limits_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}$ - Mittag-Lefler funksiyasi [2], $W^1(0,T]$ fazo esa $f\in C^1(0,T]$, $f'\in L^1(0,T]$ shartlarni qanoatlantiruvchi funksiyalar fazosini bildiradi [1].

α tartibli vazn funksiyasiga ega boʻlgan Riman-Liuvil kasr tartibli differensial operatorning berilishi quyidagicha:

$$ {}^{RL}D_w^{\alpha}g(t)=\frac{1}{w(t)}\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t\frac{w(z)g(z)}{(t-z)^{\alpha+1-n}}dz, \qquad (2)$$

bu yerda $n-1<\alpha<n$, $n\in$ □ .

ADABIYOTLAR TAHLILI VA METODLAR

Vazn funksiyasiga ega Atangana-Baleanu kasr tartibli differensial operatori Al-Refai[1] tomonidan o‘rganilgan va xossalari keltirilgan. Riman-Liuvil kasr tartibli differensial operator va Atangana-Baleanu kasr tartibli differensial operatorlarning Laplas almashtirishlari [2] kitobda berilgan.

Ushbu maqolada vazn funksiyasiga ega bo‘lgan Riman-Liuvil va Atangana-Baleanu kasr tartibli differensial operatorlar qatnashgan to‘lqin tenglamasini Laplas almashtirishi yordamida yechilgan va yechish davomida cheksiz kamayuvchi geometrik progressiya xossalaridan foydalanilgan.

NATIJA VA MUHOKAMA

Endi bevosita Koshi masalasini tadqiq etishni boshlaylik. Dastlab quyidagi kasr tartibli oddiy differensial tenglamani qaraylik:

$$ {}^{RL}D_w^{\alpha}u(t)+\lambda\,{}_cD_w^{\beta}u(t)+\mu u(t)=f(t),\quad t>0, \qquad (3)$$

bu yerda $\lambda,\mu\in$ □ , $f(t)$ - berilgan funksiya, $1<\alpha<2, 0<\beta<1$, tenglamada ishtirok etgan kasr tartibli hosilalar (1) va (2) formulalar bilan aniqlangan.

1-masala. *(3) tenglamaning*

$$ I_{0t}^{2-\alpha}u(t)\Big|_{t=0}=0,\qquad {}^{RL}D_w^{\alpha-1}u(t)\Big|_{t=0}=0 \qquad (4)$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimi topilsin.

(3) tenglamaning har ikki tomomini vazn funksiyasi $w(t)$ ga ko‘paytirib, so‘ngra $y(t)=u(t)w(t)$ va $g(t)=f(t)w(t)$ belgilashlarni kiritamiz va tenglikni ikkala tomoniga Laplas almashtirishini tatbiq qilamiz.

Kaputo ma’nosidagi Atangana-Baleanu kasr tartibli operatorni Laplas almashtirishi quyidagicha [2]:

$$ L\left\{{}_cD_w^{\beta}y(t)\right\}(s)=\frac{M(\beta)}{1-\beta}\frac{s^{\beta}Y(s)-s^{\beta-1}y(0)}{s^{\beta}+\frac{\beta}{1-\beta}}.$$

Laplas almashtirishidagi ko‘paytirish teoremasi esa quyidagicha ifodalanadi [2]:

$$ L\left\{f(t)*g(t)\right\}=L\left\{f(t)\right\}\cdot L\left\{g(t)\right\}.$$

Riman-Liuvil differensial operatorining Laplas almashtirishi esa

$$ L\left\{{}^{RL}D_{0t}^{\alpha}y(t)\right\}=s^{\alpha}L\left\{y(t)\right\}-\sum_{k=0}^{n-1}s^k\left[{}^{RL}D_{0t}^{\alpha-k-1}y(0)\right]$$

formula bilan aniqlanadi [2]. Yuqoridagi tengliklardan quyidagi natijalarga ega bo‘lamiz:

$$ s^{\alpha}Y(s)+\lambda\frac{M(\beta)}{1-\beta}\frac{s^{\beta}Y(s)}{s^{\beta}+\frac{\beta}{1-\beta}}+\mu Y(s)=G(s),$$

bu yerda $L\{y(t)\}=Y(s)$, $L\{g(t)\}=G(s)$. Bu tenglikdan avval $Y(s)$ ni topib olamiz:

$$ Y(s)=\frac{G(s)}{s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}\frac{s^{\beta}}{s^{\beta}+\frac{\beta}{1-\beta}}+\mu}$$

bu tenglikni quyidagicha yozish mumkin:

$$L^{-1}\left\{Y(s)\right\}=g(t)*L^{-1}\left\{\frac{1}{s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}\frac{s^{\beta}}{s^{\beta}+\frac{\beta}{1-\beta}}+\mu}\right\}$$

$\frac{1}{s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}\frac{s^{\beta}}{s^{\beta}+\frac{\beta}{1-\beta}}+\mu}$ ga teskari Laplas almashtirishini qilamiz [3]. Biz ushbu kasrni cheksiz kamayuvchi geometrik progressiyaga tushirish uchun tartibga solishni boshlaymiz:

$$\frac{1}{s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}\frac{s^{\beta}}{s^{\beta}+\frac{\beta}{1-\beta}}+\mu}=\frac{s^{\beta}+\frac{\beta}{1-\beta}}{\lambda\frac{M(\beta)}{1-\beta}s^{\beta}+\left(s^{\beta}+\frac{\beta}{1-\beta}\right)\left(s^{\alpha}+\mu\right)}=$$

$$=\frac{s^{\beta}+\frac{\beta}{1-\beta}}{\left(s^{\beta}+\frac{\beta}{1-\beta}\right)\left(s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)-\lambda\frac{\beta M(\beta)}{(1-\beta)^{2}}}=$$

$$=\frac{1}{s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}+\mu}\cdot\frac{1}{1-\left[\lambda\frac{\beta M(\beta)}{(1-\beta)^{2}}\frac{1}{\left(s^{\beta}+\frac{\beta}{1-\beta}\right)\left(s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)}\right]}.$$

Cheksiz kamayuvchi geometrik progressiya bo‘lishi uchun quyidagi shartni qanoatlantirishi kerak:

$$\left|\lambda\frac{\beta M(\beta)}{(1-\beta)^{2}}\frac{1}{\left(s^{\beta}+\frac{\beta}{1-\beta}\right)\left(s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)}\right|<1. \quad (5)$$

Endi, qator ko‘rinishida yozish mumkin

$$\frac{1}{s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}+\mu}\cdot\sum_{n=0}^{\infty}\frac{\left(\lambda\frac{\beta M(\beta)}{(1-\beta)^{2}}\right)^{n}}{\left(s^{\beta}+\frac{\beta}{1-\beta}\right)^{n}\left(s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)^{n}}=$$

$$=\sum_{n=0}^{\infty}\frac{\left(\lambda\frac{\beta M(\beta)}{(1-\beta)^{2}}\right)^{n}}{\left(s^{\beta}+\frac{\beta}{1-\beta}\right)^{n}\left(s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)^{n+1}}.$$

Qatordagi indekslarni siljitib olamiz:

$$\sum_{n=0}^{\infty}\frac{\left(\lambda\frac{\beta M(\beta)}{(1-\beta)^2}\right)^n}{\left(s^\beta+\frac{\beta}{1-\beta}\right)^n\left(s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)^{n+1}}=$$

$$\sum_{n=1}^{\infty}\left[\frac{\left(\lambda\frac{\beta M(\beta)}{(1-\beta)^2}\right)^n}{\left(s^\beta+\frac{\beta}{1-\beta}\right)^n\left(s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)^{n+1}}\right]+\frac{1}{s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu}=$$

$$\sum_{n=0}^{\infty}\left[\frac{\left(\lambda\frac{\beta M(\beta)}{(1-\beta)^2}\right)^{n+1}}{\left(s^\beta+\frac{\beta}{1-\beta}\right)^{n+1}\left(s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)^{n+2}}\right]+\frac{1}{s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu}.$$

Teskari Laplas almashtirishini qo‘llab, quyidagi natijani olamiz:

$$L^{-1}\left\{\sum_{n=0}^{\infty}\left[\frac{\left(\lambda\frac{\beta M(\beta)}{(1-\beta)^2}\right)^{n+1}}{\left(s^\beta+\frac{\beta}{1-\beta}\right)^{n+1}\left(s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)^{n+2}}\right]+\frac{1}{s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu}\right\}=$$

$$=L^{-1}\left\{\sum_{n=0}^{\infty}\left[\frac{\left(\lambda\frac{\beta M(\beta)}{(1-\beta)^2}\right)^{n+1}}{\left(s^\beta+\frac{\beta}{1-\beta}\right)^{n+1}\left(s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)^{n+2}}\right]\right\}+L^{-1}\left\{\frac{1}{s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu}\right\}=$$

$$=\sum_{n=0}^{\infty}L^{-1}\left\{\left[\frac{\left(\lambda\frac{\beta M(\beta)}{(1-\beta)^2}\right)^{n+1}}{\left(s^\beta+\frac{\beta}{1-\beta}\right)^{n+1}\left(s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)^{n+2}}\right]\right\}+L^{-1}\left\{\frac{1}{s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu}\right\}=$$

$$=\sum_{n=0}^{\infty}\left[\left(\lambda\frac{\beta M(\beta)}{(1-\beta)^2}\right)^{n+1}L^{-1}\left\{\frac{1}{\left(s^\beta+\frac{\beta}{1-\beta}\right)^{n+1}}\cdot\frac{1}{\left(s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)^{n+2}}\right\}\right]+L^{-1}\left\{\frac{1}{s^\alpha+\lambda\frac{M(\beta)}{1-\beta}+\mu}\right\}$$

Laplas almashtirishini ko‘paytirish teoremasidan foydalanamiz:

$$\sum_{n=0}^{\infty}\left[\left(\lambda\frac{\beta M(\beta)}{(1-\beta)^2}\right)^{n+1}L^{-1}\left\{\frac{1}{\left(s^{\beta}+\frac{\beta}{1-\beta}\right)^{n+1}}\right\}*L^{-1}\left\{\frac{1}{\left(s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)^{n+2}}\right\}\right]+$$

$$+t^{\alpha-1}E_{\alpha,\alpha}\left(-\left[\lambda\frac{M(\beta)}{1-\beta}+\mu\right]t^{\alpha}\right).$$

Mittag-Leffler funksiyaning Laplas almashtirishi quyidagicha bo‘ladi:

$$L\left\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right\}(s)=\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}.$$

Ushbu almashtirishdan foydalanib, quyidagi natijani olamiz:

$$L^{-1}\{\cdot\}=\sum_{n=0}^{\infty}\left(\lambda\frac{\beta M(\beta)}{(1-\beta)^2}\right)^{n+1}\left[\left(\frac{t^{\beta(n+1)-1}E_{\beta,\beta}^{(n)}\left(-\frac{\beta}{1-\beta}t^{\beta}\right)}{n!}\right)*\left(\frac{t^{\alpha(n+2)-1}E_{\alpha,\alpha}^{(n+1)}\left(-\left[\lambda\frac{M(\beta)}{1-\beta}+\mu\right]t^{\alpha}\right)}{(n+1)!}\right)\right]+$$

$$+t^{\alpha-1}E_{\alpha,\alpha}\left(-\left[\lambda\frac{M(\beta)}{1-\beta}+\mu\right]t^{\alpha}\right)$$

$$=\sum_{n=0}^{\infty}c^{n+1}\left[\left(\frac{t^{\beta(n+1)-1}}{n!}\cdot\sum_{l=0}^{\infty}\frac{b^l t^{\beta l}\Gamma(n+l+1)}{\Gamma(\beta(l+n+1))l!}\right)*\left(\frac{t^{\alpha(n+2)-1}}{(n+1)!}\cdot\sum_{m=0}^{\infty}\frac{a^m t^{\alpha m}\Gamma(n+m+2)}{\Gamma(\alpha(m+n+2))m!}\right)\right]+$$

$$+t^{\alpha-1}E_{\alpha,\alpha}\left(-\left[\lambda\frac{M(\beta)}{1-\beta}+\mu\right]t^{\alpha}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{c^{n+1}b^l a^m\Gamma(n+l+1)\Gamma(n+m+2)}{(n+1)!n!l!m!\Gamma(\beta(l+n+1))\Gamma(\alpha(m+n+2))}\int_0^t x^{\alpha(n+m+2)-1}(t-x)^{\beta(l+n+1)-1}dx$$

$$+t^{\alpha-1}E_{\alpha,\alpha}\left(at^{\alpha}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\left[\binom{n+m+1}{m}\binom{n+l}{l}\frac{c^{n+1}b^l a^m t^{\alpha(m+n+2)+\beta(l+n+1)-1}}{(n+1)!n!l!m!\Gamma(\alpha(m+n+2)+\beta(l+n+1))}\right]+$$

$$+t^{\alpha-1}E_{\alpha,\alpha}\left(at^{\alpha}\right),$$

bu yerda $a=-\lambda\frac{M(\beta)}{1-\beta}-\mu,\ b=-\frac{\beta}{1-\beta},\ c=\lambda\frac{\beta M(\beta)}{(1-\beta)^2}.$

Yuqoridagi hisoblashlarni hisobga olib quyidagini hosil qilamiz:

$$y(t)=g(t)*L^{-1}\left\{\frac{1}{s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}\frac{s^{\beta}}{s^{\beta}+\frac{\beta}{1-\beta}}+\mu}\right\}=g(t)*\varphi(t)\text{'}\qquad(6)$$

bu yerda

$$\varphi(t)=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\left[\binom{n+m+1}{m}\binom{n+l}{l}\frac{c^{n+1}b^l a^m t^{\alpha(m+n+2)+\beta(l+n+1)-1}}{(n+1)!n!l!m!\Gamma(\alpha(m+n+2)+\beta(l+n+1))}\right]+$$

$$+t^{\alpha-1}E_{\alpha,\alpha}\left(at^{\alpha}\right).$$

Kiritilgan $y(t)=u(t)w(t)$ va $g(t)=f(t)w(t)$ belgilashlarga ko'ra Koshi masalasining yechimi topiladi. Demak, quyidagi teorema o'rinli:

1-teorema. Agar $u(t)\in W^1(0,T]$ va (5) shart bajarilsa, (3)-(4) Koshi masalasining yechimi (6) formula bilan ifodalanadi.

Quyidagi uchta tengsizlikni ko'rib chiqamiz, bu yerda $\tilde{c}$- musbat haqiqiy o'qdagi gamma-funksiyasining minimal qiymati va $x, y\in R$:

$$x, y>\tilde{c}\Rightarrow \frac{1}{\Gamma(x+y)}\le\frac{1}{\Gamma(x)\Gamma(y)} \tag{7}$$

$$x>\tilde{c},\ \tilde{c}>y>0\Rightarrow \frac{1}{\Gamma(x+y)}\le\frac{1}{\Gamma(x)} \tag{8}$$

$$\tilde{c}>x>0 \text{ va } \tilde{c}>y>0\ \Rightarrow \frac{1}{\Gamma(x+y)}\le\frac{1}{\Gamma(\tilde{c})} \tag{9}$$

$\tilde{c}$ ta'rifi bo'yicha musbat haqiqiy o'qning minimal nuqtasi, (9) aniq to'g'ri, (1) ifoda $x>\tilde{c}$ uchun ekanligi kelib chiqadi, $\Gamma(x)$- o'suvchi funksiya. Va nihoyat, bizda (7) natija [4]. Bizdagi [4] dan hamma $x, y>1$ uchun $\mathrm{B}(x,y)\le\frac{1}{xy}$ mavjud. Chunki $x, y>1$, bizda $\frac{1}{xy}\le 1$ mavjud

$$\mathrm{B}(x,y)\le\frac{1}{xy}\le 1$$

va

$$\mathrm{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

$\Gamma(x+y)\ge\Gamma(x)\Gamma(y)$ hamma $x, y>1$ lar uchun o'rinli.

E'tibor bering, ushbu holatlarning har qandayi bizda mavjud:

$$\frac{1}{\Gamma(x+y)}\le\frac{\prod_{j\in\{x,y\}\&\ j<\tilde{c}}\frac{\Gamma(j)}{\Gamma(\tilde{c})}}{\Gamma(x)\Gamma(y)}.$$

Uchta holatni berish uchun ushbu tengsizlikni takrorlash mumkin:

$$\frac{1}{\Gamma(x+y+z)}\le\frac{\prod_{j\in\{x,y+z\}\&\ j<\tilde{c}}\frac{\Gamma(j)}{\Gamma(\tilde{c})}}{\Gamma(x)\Gamma(y+z)}\le\frac{\prod_{j\in\{x,y,z\}\&\ j<\tilde{c}}\frac{\Gamma(j)}{\Gamma(\tilde{c})}}{\Gamma(x)\Gamma(y)\Gamma(z)}.$$

Qo'shimcha ravishda, bu $\prod_{j\in\{x,y,z\}\&\ j<\tilde{c}}\frac{\Gamma(j)}{\Gamma(\tilde{c})}>1$ biz quyidagi yig'indini ko'rsatilgandek bog'lashimiz mumkinligini anglatadi:

$$\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\left|\frac{c_1^n c_2^m c_3^l t^{a_1 n+a_2 m+a_3 l+a_4}}{\Gamma(b_1 n+b_2 m+b_3 l+b_4)}\right|\le\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\left|\frac{C c_1^n c_2^m c_3^l t^{a_1 n+a_2 m+a_3 l+a_4}}{\Gamma\left(b_1 n+\frac{b_4}{3}\right)\Gamma\left(b_2 m+\frac{b_4}{3}\right)\Gamma\left(b_3 l+\frac{b_4}{3}\right)}\right|\le$$

$$\leq Ct^{a_4}\sum_{n=0}^{\infty}\left|\frac{c_1^n t^{a_1 n}}{\Gamma\left(b_1 n+\frac{b_4}{3}\right)}\right|\sum_{m=0}^{\infty}\left|\frac{c_2^m t^{a_2 m}}{\Gamma\left(b_2 m+\frac{b_4}{3}\right)}\right|\sum_{l=0}^{\infty}\left|\frac{c_3^l t^{a_3 l}}{\Gamma\left(b_3 l+\frac{b_4}{3}\right)}\right|=$$

$$=Ct^{a_4}E_{b_1,\frac{b_4}{3}}\left(\left|c_1t^{a_1}\right|\right)E_{b_2,\frac{b_4}{3}}\left(\left|c_2t^{a_2}\right|\right)E_{b_3,\frac{b_4}{3}}\left(\left|c_3t^{a_3}\right|\right).$$

Endi yuqoridagilarni hisobga olib, $\varphi(t)$ funksiyani chegarasini aniqlash uchun foydalanishimiz mumkin,

$$\binom{n}{k}\leq 2^n$$

$$\varphi(t)=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\left[\binom{n+m+1}{m}\binom{n+l}{l}\frac{c^{n+1}b^l a^m t^{\alpha(m+n+2)+\beta(l+n+1)-1}}{(n+1)!n!l!m!\Gamma\left(\alpha(m+n+2)+\beta(l+n+1)\right)}\right]\leq$$

$$\leq Ct^{2\alpha+\beta-1}E_{\alpha,\frac{2\alpha+\beta}{3}}\left(\left|2at^{\alpha}\right|\right)E_{\beta,\frac{2\alpha+\beta}{3}}\left(\left|2bt^{\beta}\right|\right)E_{\alpha+\beta,\frac{2\alpha+\beta}{3}}\left(\left|4ct^{\alpha+\beta}\right|\right).$$

Laplasning teskari shartlari:

Bundan tashqari, mavjudlikni tekshirish uchun, biz Laplasni teskari tomonga olganimizda hech qanday qarama-qarshilik yo‘qligini tekshirishimiz kerak. Quyidagi tengsizlikni qanoatlantiradigan haqiqiy s soni mavjud bo‘lishi kerak:

$$\left|\lambda\frac{\beta M(\beta)}{(1-\beta)^2}\frac{1}{\left(s^{\beta}+\frac{\beta}{1-\beta}\right)\left(s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)}\right|<1.$$

Soddalashtirsak,

$$\left|\lambda\frac{\beta M(\beta)}{(1-\beta)^2}\right|<\left|\left(s^{\beta}+\frac{\beta}{1-\beta}\right)\left(s^{\alpha}+\lambda\frac{M(\beta)}{1-\beta}+\mu\right)\right|$$

[5] dagi teskari Laplas almashtirishidan foydalanamiz:

$$s>\max\left\{\left|\frac{\beta}{1-\beta}\right|^{\frac{1}{\beta}},\left|\lambda\frac{M(\beta)}{1-\beta}-\mu\right|^{\frac{1}{\alpha}}\right\}.$$

Endi, biz $u(x,t)$ aniqlashning masalasini ko‘rib chiqamiz, $\Omega=\left\{(x,t):0<x<1,\,0<t<T\right\}$ sohada

$$^{RL}D_w^{\alpha}u(x,t)+\lambda\cdot{}^{AB}_{w}D_{0t}^{\beta}u(x,t)-u_{xx}(x,t)=f(x,t),\qquad(10)$$

tenglamani qaraymiz, bu yerda $f(x,t)$ berilgan funksiya. (10) tenglama uchun Ω sohada quyidagi masalani ko‘ramiz.

2-masala. *Shunday $u(x,t)$ yechimi topilsinki, u Ω sohada (10) tenglamani*

$$u(x,0)=0,\ 0\leq x\leq 1,\qquad I_{0t}^{\alpha-1}u(x,t)\Big|_{t=0}=0,\ 0<x<1,\qquad(11)$$

Boshlang‘ich shart va

$$u(0,t)=0,\quad u(1,t)=0,\ 0\leq t\leq 1\qquad(12)$$

chegaraviy shartlarni qanoatlantirsin. Bu yerda $\lambda,\alpha,\beta\in R$, $1<\alpha\leq 2,\quad 0<\beta<1$.

(10)-(12) masalani yechishda o‘zgaruvchilarni ajratish usulidan foydalanamiz. Faraz qilaylik,

$$u(x,t) = X(x)U(t)$$

bo‘lsin. U holda (3.2.10) tenglamani qayta yozishimiz mumkin,

$$X(x)\,{}^{RL}D_w^{\alpha}U(t) + X(x)\lambda \cdot {}_{w}^{AB}D_{0t}^{\beta}U(t) - X''(x)U(t) = 0$$

$$\frac{{}^{RL}D_w^{\alpha}U(t) + \lambda \cdot {}_{w}^{AB}D_{0t}^{\beta}U(t)}{U(t)} = \frac{X''(x)}{X(x)} = -\xi$$

Shunday qilib, biz quyidagi spektral masalani yechamiz:

$$\begin{cases} X'' + \xi X = o \\ X(0) = 0,\ X(1) = 0 \end{cases} \tag{13}$$

yuqoridagi tenglamaning umumiy yechimi

$$X(x) = C_1 \cos\sqrt{A}x + C_2 \sin\sqrt{A}x$$

ko‘rinishda bo‘lib, bu yechimni (3.2.13) shartga bo‘ysintirsak,

$$\xi_k = \pi^2 k^2,\ \ k = 1,2,3,...$$

xos sonlarga, ularga mos

$$X_k = \sin(k\pi x), \qquad k = 1,2,3,...$$

xos funksiyalarga ega bo‘lamiz. $u(x,t)$ va $f(x,t)$ funksiyalarni quyidagi ko‘rinishda yozish mumkin:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\sin k\pi x, \qquad f(x,t) = \sum_{k=1}^{\infty} f_k(t)\sin k\pi x, \tag{14}$$

bu yerda

$$f_k(t) = 2\int_0^1 f(x,t)\sin k\pi x\,dx$$

va $u_k(t)$ keyinchalik aniqlanadigan noma’lum funksiya.

Endi, (10) tenglamaga (14) tenglikni olib borib qo‘yamiz,

$${}^{RL}D_w^{\alpha}\sum_{k=1}^{\infty} u_k(t)\sin k\pi x + \lambda\,{}_{w}^{AB}D_{0t}^{\beta}\sum_{k=1}^{\infty} u_k(t)\sin k\pi x - \frac{\partial^2}{\partial x^2}\sum_{k=1}^{\infty} u_k(t)\sin k\pi x =$$

$$= \sum_{k=1}^{\infty} f_k(t)\sin k\pi x$$

yoki

$$\sum_{k=1}^{\infty}\left[{}^{RL}D_w^{\alpha}u_k(t) + \lambda\,{}_{w}^{AB}D_{0t}^{\beta}u_k(t) + (k\pi)^2 u_k(t)\right]\sin k\pi x = \sum_{k=1}^{\infty} f_k(t)\sin k\pi x.$$

U holda, L_2 sinfda $\{\sin k\pi x\}$ ning ortoganalligi tufayli, yechim topish kifoya

$${}^{RL}D_w^{\alpha}u_k(t) + \lambda\,{}_{w}^{AB}D_{0t}^{\beta}u_k(t) + (k\pi)^2 u_k(t) = f_k(t).$$

Bu $u(x,t)$ ning to‘liq yechimini topishga yordam beradi:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\sin k\pi x, \tag{15}$$

$$u_k(t)=\frac{1}{w(t)}\int_0^t f_k(s)w(s)\varphi(t-s)ds$$

bu yerda

$$\varphi(t)=\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\left[\binom{n+m+1}{m}\binom{n+l}{l}\frac{c^{n+1}b^l a^m t^{\alpha(m+n+2)+\beta(l+n+1)-1}}{(n+1)!n!l!m!\Gamma(\alpha(m+n+2)+\beta(l+n+1))}\right]+$$

$$+t^{\alpha-1}E_{\alpha,\alpha}\left(at^{\alpha}\right),$$

$$a=-\lambda\frac{M(\beta)}{1-\beta}-\mu,\ b=-\frac{\beta}{1-\beta},\ c=\lambda\frac{\beta M(\beta)}{(1-\beta)^2},$$

$$f_k(t)=2\int_0^1 f(x,t)\sin k\pi x\,dx.$$

2-teorema.

Agar $u_k(t)\in W^1(0,T]$ va (5) shart bajarilsa, 2-masalaning yechimi (15) formula bilan ifodalanadi.

XULOSA

Ushbu maqolada vazn funksiyasiga ega bo‘lgan Riman-Liuvil va Atangana-Baleanu kasr tartibli differensial operatorlar qatnashgan to‘lqin tenglamasi ko‘rildi. Vazn funksiyasiga ega bo‘lgan Riman-Liuvil va Atangana-Baleanu kasr tartibli differensial operatorlar qatnashgan differensial tenglama uchun Koshi masalasini yechimi topildi(6). Keyin ushbu yechimdan foydalanib, operatorlar qatnashgan to‘lqin tenglamasi uchun chegaraviy masala yechimi topilgan.

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